



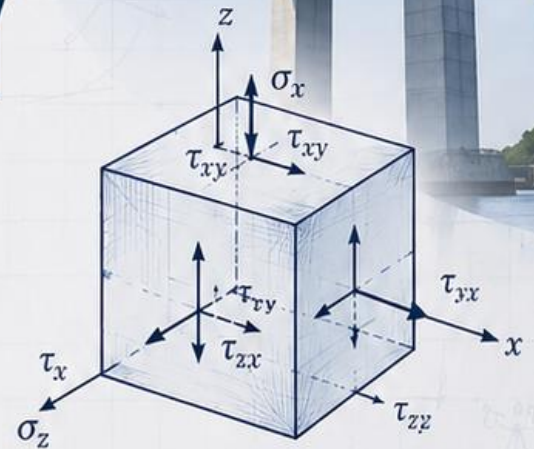
PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION
AND SCIENTIFIC RESEARCH

ZIANE ACHOUR UNIVERSITY – DJELFA
FACULTY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF CIVIL ENGINEERING

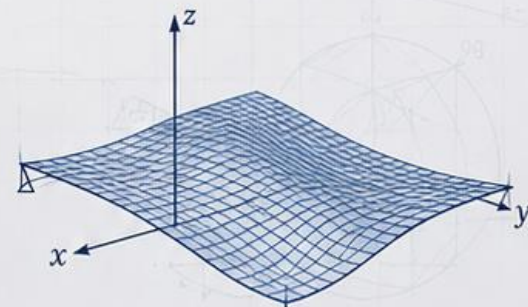
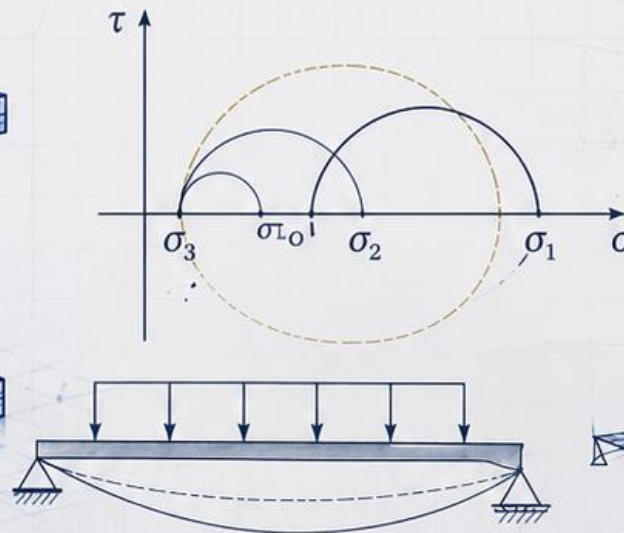
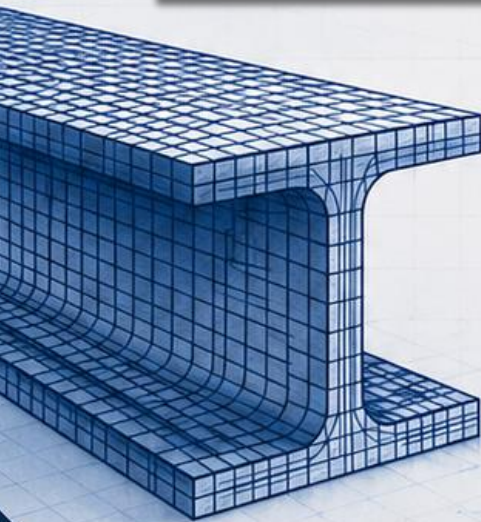
Handout of ELASTICITY

THEORY – ANALYSIS – APPLICATIONS

FOR THE FIRST-YEAR MASTER'S OF CIVIL
ENGINEERING
(SPECIALTY: STRUCTURES)



Prepared by
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FOUNDATIONS

Stress, Strain and
Elasticity Theory



ANALYSIS

Equations, Methods
and Problem Solving



APPLICATIONS

2025/2026
Beams, Plates and
Structural Elements



ENGINEERING TOOLS

Modeling and Use of
Engineering Software



SEMESTER

2

CREDITS: 6
COEFFICIENT: 3

Course Academic and Pedagogical Specifications

Course (Module): Elasticity

Semester: 2

Teaching Unit: UEF 1.2.1

Total Hours (VHS): 67h30 (Lectures: 3h00, Tutorials: 1h30)

Credits: 6

Coefficient: 3

Assessment Method: Continuous Assessment: 40% and Final Exam: 60%

Recommended Prerequisites: Basic knowledge of Mathematics and Strength of Materials

Course Objectives:

This course aims to provide students with fundamental analytical and computational methods for studying the mechanical behavior of elastic bodies and structural elements. It enables students to understand stress, strain, and constitutive relations, and to apply elasticity theory in solving engineering problems. The course develops the ability to analyze and design structures in a rational and safe manner under different loading conditions. It also strengthens problem-solving skills through mathematical modeling of continuum mechanics problems. Students will acquire the theoretical foundations necessary for advanced structural analysis and numerical methods, particularly the finite element method. In addition, the course prepares students to effectively use engineering software for simulation and analysis. Ultimately, the course seeks to build a solid foundation for practical applications in civil and structural engineering.

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General Introduction

Context and Motivation

The mechanical behavior of solid bodies under external loading constitutes one of the most fundamental and practically significant areas of engineering science. From the analysis of civil infrastructure to the design of aerospace structures, the ability to predict how materials deform, develop internal stresses, and ultimately reach their limits of safe operation is central to the work of structural and mechanical engineers. Elasticity theory provides the rigorous mathematical and physical framework that underpins all such analyses, offering tools to describe reversible deformation, stress distributions, and the recovery of shape upon removal of loads.

The present course handout is designed for first-year Master's students in Civil Engineering, specializing in Structures, at Ziane Achour University of Djelfa. It responds to the need for a unified, self-contained, and pedagogically accessible treatment of elasticity theory at an advanced undergraduate and early graduate level. The course spans two theoretical pillars: the mathematical formalism necessary to express field equations in compact and invariant form, and the physical insight required to interpret solutions in terms of real engineering phenomena.

Scope and Objectives

The primary objective of this course is to equip students with both the analytical and conceptual tools needed to study the mechanical behavior of elastic bodies and structural elements. Students are expected to gain a deep understanding of the interrelated concepts of stress, strain, and constitutive laws, and to develop the capacity to formulate and solve elasticity problems in two and three dimensions. The course also prepares students for the study of advanced structural analysis techniques, in particular the finite element method, which is now indispensable in professional engineering practice.

Beyond the acquisition of analytical skills, this course aims to cultivate a rational and critical approach to structural modeling. By working through the derivation of governing equations from first principles, students learn to distinguish between exact solutions and engineering approximations, to recognize the limitations of classical theories, and to apply simplifying assumptions judiciously in practical contexts. This capacity for theoretical rigor combined with engineering judgment is the hallmark of a well-trained structural engineer.

Structure of the Course

The course is organized into eight chapters, each addressing a distinct but interconnected aspect of elasticity theory. The first chapter introduces the foundational concepts of continuum mechanics, including the classification of elastic materials, Hooke's Law in its elementary form, and the mathematical tools of index notation and tensor algebra. These tools are essential for the compact and invariant expression of field equations throughout the course.

Chapters Two and Three are devoted to the theory of stress and strain states, respectively. In Chapter Two, the Cauchy stress tensor is introduced, including its physical interpretation, principal stresses, stress invariants, the differential equations of equilibrium, and Mohr's circle for graphical representation. Chapter Three treats the kinematics of deformation, covering the Green–Lagrange and linearized strain tensors, the small perturbation hypothesis, strain–

displacement relations in multiple coordinate systems, principal strains, compatibility equations, and experimental measurement techniques.

Chapter Four presents the constitutive laws governing the elastic response of materials, including the generalized Hooke's law in one, two, and three dimensions, the role of Lamé and engineering elastic constants, orthotropy and anisotropy, thermoelastic effects, and the strain energy concept. Chapter Five consolidates these developments into the general equations of linear elasticity, specifically the Navier–Cauchy displacement equations and the Beltrami–Michell stress compatibility equations, together with Saint-Venant's principle.

The remaining three chapters shift the focus toward structural applications. Chapter Six develops the theory of plane elasticity, including plane stress and plane strain formulations and the Airy stress function approach for solving two-dimensional problems analytically. Chapter Seven addresses beam bending through both the Euler–Bernoulli and Timoshenko beam theories, comparing their assumptions, governing equations, and ranges of applicability. Chapter Eight introduces the study of thin plates using the Kirchhoff–Love model, covering plate kinematics, constitutive behavior, equilibrium equations, the governing biharmonic equation, and strain energy formulations.

Prerequisites and Learning Outcomes

This course assumes a solid background in mathematics, particularly in linear algebra, differential equations, and vector calculus, as well as a working knowledge of Strength of Materials at the undergraduate level. Upon successful completion of the course, students will be able to apply index notation and tensor calculus to formulate elasticity problems; analyze stress and strain states including principal values, invariants, and graphical representations; use constitutive laws to relate stress and strain for isotropic, orthotropic, and anisotropic materials; derive and apply the Navier–Cauchy and Beltrami–Michell equations; solve plane elasticity problems using Airy stress functions; analyze beam bending using Euler–Bernoulli and Timoshenko theories; and study the bending behavior of thin elastic plates.

Together, these competencies form a comprehensive foundation for graduate-level work in structural mechanics, numerical methods, and advanced material design, positioning students to contribute meaningfully to research and professional practice in civil and structural engineering.

Chapter 1: Introduction to Elasticity Theory

- 1. Introduction**
- 2. General concepts of elasticity**
- 3. Index notation**
- 4. Fundamentals of Vectors**
- 5. Conclusion**

1. Introduction

Continuum mechanics is a branch of mechanics whose objective is the study of the dynamic (motion) or static behavior of continuous media. This field can be considered the engineering science par excellence, as it allows us to describe and understand the material world around us and the common phenomena that occur within it. It deals with deformable solids, fluids (liquids and gases), and materials exhibiting intermediate behaviors.

Continuum mechanics includes elasticity, plasticity, fluid mechanics, and strength of materials (SM). Elasticity theory studies the behavior of homogeneous, isotropic, perfectly elastic, deformable solids with a linear constitutive law. An additional simplifying assumption is made: displacements are small. Elasticity theory examines displacements, strains, and stresses in a solid subjected to external forces.

This chapter begins with the general concepts of elasticity, where the basic assumptions, physical meaning, and fundamental principles governing the elastic behavior of materials are presented. This section aims to clarify how deformable solids respond to external loads under elastic conditions.

Then, a mathematical review is provided to recall the essential tools needed to develop the theory. This includes vector and tensor algebra, differential operators, and other mathematical concepts frequently used in continuum mechanics.

The chapter continues with index notation, which is introduced as a powerful and compact way to express tensorial equations. This notation simplifies complex derivations and makes the formulation of elasticity equations more systematic and easier to manipulate.

Finally, a conclusion summarizes the key points of the chapter and prepares the reader for the next developments in the study of elasticity.

2. General concepts of elasticity

2.1 Definition of Elasticity

Elasticity is the property of a material to return to its original shape and size after the external forces causing deformation are removed. In simple words, if you stretch or compress an elastic material and then release it, it goes back to its initial state.

Elastic behavior is reversible, meaning no permanent deformation remains after unloading (as long as the material does not exceed its elastic limit).

2.2 Types of Deformation in Solids

When a solid is loaded, it can deform in different ways:

- Tensile or compressive deformation: The material becomes longer (tension) or shorter (compression).
- Shear deformation: The shape changes without a significant change in volume, like sliding layers over each other.
- Volumetric deformation: The volume of the material changes due to pressure (compression or expansion).

These deformation types can occur separately or together in real structures.

2.3 Stress and Strain Concepts (Physical View)

Stress represents internal forces inside a material due to external loading. It is force per unit area.

Example: pulling a rod creates internal resistance forces.

Strain represents how much the material deforms compared to its original size. It is a measure of deformation.

Simply:

Stress = “cause” (forces applied)

Strain = “effect” (deformation produced)

Elasticity studies how stress leads to strain.

2.4 Assumptions of Linear Elasticity

To simplify real material behavior, we use some assumptions:

- Small deformations: The changes in shape are very small compared to the original size.
- Linear behavior: Stress is directly proportional to strain (doubling force doubles deformation).
- Homogeneous material: Material properties are the same everywhere.
- Isotropic material (often assumed): Properties are the same in all directions.
- Continuity: Physical properties vary continuously from one point to another of the studied medium.

These assumptions make mathematical modeling much easier.

2.5 Hooke's Law (Basic Formulation)

Hooke's Law describes the simplest relationship in elasticity (Stress is proportional to strain).

In 1D form, if you stretch a spring or bar, the deformation increases linearly with the applied force. This law is valid only within the elastic limit of the material.

It introduces the concept of elastic constants, which define a material's stiffness.

2.6 Elastic Material Properties

These are constants that describe how stiff or flexible a material is:

- Young's Modulus (E): Measures stiffness in tension or compression.
- Poisson's Ratio (ν): Describes how a material contracts in one direction when stretched in another.
- Shear Modulus (G): Measures resistance to shape change under shear forces.
- Bulk Modulus (K): Measures resistance to volume change under pressure.

Each material has its own values of these properties.

2.7 Classification of Elastic Materials

Materials can be classified based on their behavior:

- Isotropic materials: Same properties in all directions (steel, aluminum).
- Anisotropic materials: Properties depend on direction (wood, composites).
- Homogeneous materials: Same properties at every point.
- Non-homogeneous materials: Properties vary from point to point (functionally graded materials).

This classification is important in engineering modeling.

2.8 Energy Concept in Elasticity (optional but strong)

When a material is deformed, it stores energy inside it called strain energy.

- This energy comes from the work done by external forces.
- If the material is elastic, this energy is fully recovered when the load is removed.

Think of it like stretching a rubber band: the energy you put in is stored and released when it returns to its original shape. This concept is very important in structural analysis and failure prediction.

3. Index notation

In Elasticity, index notation, also called tensor notation or Einstein notation, is widely used to express mathematical relations in a compact and elegant form. This notation simplifies vector and tensor operations and provides a coordinate-independent formulation of physical laws. Index notation is particularly useful in elasticity theory because stress, strain, displacement, and constitutive equations involve multiple components in three-dimensional space.

3.1 Components of Vectors and Tensors

A vector in a three-dimensional Cartesian coordinate system is written as:

$$\vec{A} = a_i \vec{e}_i$$

where:

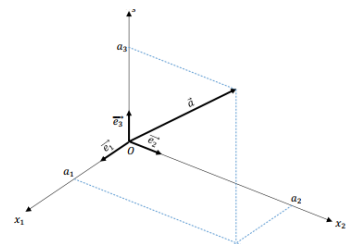
- a_i are the components of the vector,
- $\vec{e}_i$ are the basis vectors,
- the index i takes the values 1,2,3

Similarly, a second-order tensor is represented by

$$T = T_{ij} \vec{e}_i \otimes \vec{e}_j$$

where

- T_{ij} are the tensor components.



- $\vec{e}_i$ and $\vec{e}_j$ are the basis vectors,
- $\otimes$ denotes the tensor product,

The indices i and j vary from 1 to 3 in a three-dimensional Cartesian coordinate system.

$$\begin{aligned} T = & T_{11}\vec{e}_1 \otimes \vec{e}_1 + T_{12}\vec{e}_1 \otimes \vec{e}_2 + T_{13}\vec{e}_1 \otimes \vec{e}_3 \\ & + T_{21}\vec{e}_2 \otimes \vec{e}_1 + T_{22}\vec{e}_2 \otimes \vec{e}_2 + T_{23}\vec{e}_2 \otimes \vec{e}_3 \\ & + T_{31}\vec{e}_3 \otimes \vec{e}_1 + T_{32}\vec{e}_3 \otimes \vec{e}_2 + T_{33}\vec{e}_3 \otimes \vec{e}_3 \end{aligned}$$

This expansion shows that a second-order tensor is formed by combining all possible tensor products between basis vectors.

3.2 Einstein Summation Convention

The Einstein summation convention states that whenever an index appears twice in a single term, summation over that index is automatically implied.

For example:

$$A_i B_i = \sum_{i=1}^3 A_i B_i$$

Thus:

$$A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

This convention considerably reduces the complexity of mathematical expressions in continuum mechanics.

The tensor expression may be expanded as:

3.3 Free and Dummy Indices

Indices in tensor notation are classified into two categories:

- A free index appears only once in each term of an equation and determines the component of the resulting quantity.

$$C_i = A_i + B_i$$

The index i is free because it appears once on each side of the equation.

- A dummy index appears twice in a term and implies summation. For example, $A_i B_i$

Dummy indices may be replaced by another symbol without changing the meaning:

$$A_i B_i = A_j B_j$$

3.4 Kronecker Delta

The Kronecker delta is a second-order identity tensor defined by:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

It possesses the important property:

$$\delta_{ij} A_j = A_i$$

The Kronecker delta is frequently used in tensor manipulations and constitutive equations.

Example

Let: $A_j = (7, -2, 4)$

Then:

$$\delta_{ij} A_j = \begin{cases} A_1 & i = 1 \\ A_2 & i = 2 \\ A_3 & i = 3 \end{cases}$$

Therefore:

$$\delta_{ij}A_j = (7, -2, 4)$$

The Kronecker delta acts as the identity operator.

3.5 Levi-Civita Symbol

The Levi-Civita symbol is a third-order antisymmetric tensor denoted by ε_{ijk} . It is defined as:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{for even permutations of (1,2,3)} \\ -1 & \text{for odd permutations of (1,2,3)} \\ 0 & \text{if any two indices are equal} \end{cases}$$

Example

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) = (1, 2, 3); (2, 3, 1); (3, 1, 2) \\ -1 & \text{if } (i, j, k) = (3, 2, 1); (2, 1, 3); (1, 3, 2) \\ 0 & \text{if } i=j, j=k; k=i \end{cases}$$

It is commonly used in cross-product operations:

3.7 Advantages of Index Notation

Index notation offers several advantages in elasticity and continuum mechanics:

- compact representation of equations,
- simplification of vector and tensor operations,
- coordinate-independent formulation,
- easier manipulation of complex tensor equations,
- efficient derivation of constitutive and equilibrium equations.

This notation forms the mathematical foundation for the study of stress analysis, strain theory, and material behavior in elasticity.

4. Fundamentals of Vectors

Continuum mechanics makes extensive use of scalar, vector, and tensor fields. Through vector and tensor operations, the equations governing mechanics can be expressed in an intrinsic and compact mathematical form. A physical quantity may be represented mathematically by [3]:

- A scalar quantity (density, temperature, pollutant concentration, etc.).
- A vector quantity (acceleration, body forces, body moments, etc.).
- A second-order tensor (strain, stress, etc.).
- A higher-order tensor, such as the elasticity tensor, which is a fourth-order tensor.

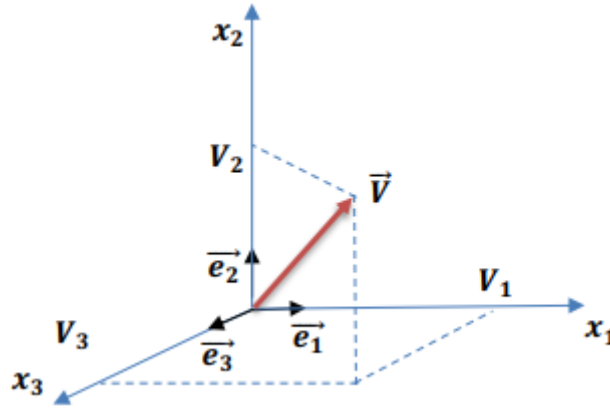
The use of tensors in continuum mechanics provides a rigorous framework for describing the mechanical behavior of materials and physical systems independently of the chosen coordinate system. This mathematical formalism is essential in solid mechanics, fluid mechanics, elasticity theory, and computational mechanics.

4.1 Vector

In a three-dimensional Euclidean space over the field of real numbers $\mathbb{R}^3$, we define three unit vectors denoted by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, associated respectively with the coordinate axes x_1 , x_2 , and x_3 . Let

$B(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis. A vector $\mathbf{V}$ is represented by its components V_1 , V_2 , and V_3

$$\vec{V} = V_1\vec{e}_1 + V_2\vec{e}_2 + V_3\vec{e}_3 = \sum_{i=1}^3 V_i\vec{e}_i$$



The basis vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ have the respective components:

$$\vec{e}_1 = (1, 0, 0)$$

$$\vec{e}_2 = (0, 1, 0)$$

$$\vec{e}_3 = (0, 0, 1)$$

Using Einstein's summation convention, repeated indices imply summation over the index range. This notation simplifies tensor and vector expressions commonly used in continuum mechanics and mathematical physics.

In matrix notation, the vector can be written as:

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

The transpose of the vector is expressed as:

$$V^T = [V_1 \quad V_2 \quad V_3]$$

The magnitude (or norm) of a vector $\vec{V}$ with components (V_1, V_2, V_3) relative to the orthonormal coordinate system $(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ is denoted by $\|\vec{V}\|$ and is defined by:

$$\|\vec{V}\| = \sqrt{V_1^2 + V_2^2 + V_3^2}$$

The vector norm represents the geometric length or magnitude of the vector in space. It is a fundamental quantity in mechanics and physics because it characterizes the intensity of physical quantities such as force, velocity, and displacement

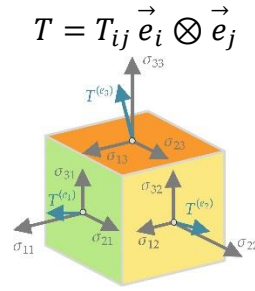
4.2 Tensors

In continuum mechanics, tensors are mathematical entities used to represent physical quantities that possess magnitude and direction in multiple dimensions. Unlike scalars, which are represented by a single value, or vectors, which are represented by one-dimensional components, tensors provide a more general mathematical framework capable of describing complex physical phenomena such as stress, strain, deformation, and material behavior.

A tensor of second order in a three-dimensional Euclidean space is represented by a set of nine components relative to an orthonormal basis. These components are arranged in matrix form as follows:

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

Using Einstein's summation convention, a second-order tensor can also be written in compact form as:



In elasticity, tensors are essential for describing internal mechanical states within deformable bodies.

The stress tensor represents internal forces acting on material surfaces, while the strain tensor characterizes deformation. Tensor notation allows the governing equations of continuum mechanics to be expressed in compact and invariant form independently of the coordinate system.

4.3 Vector Operations

Let $\vec{A}$ and $\vec{B}$ be two vectors defined in a three-dimensional Euclidean space:

$$\vec{A} = A_i \vec{e}_i, \quad \vec{B} = B_i \vec{e}_i$$

where $\vec{e}_i$ denotes the basis vectors and A_i, B_i are the vector components.

4.3.1 Vector Addition and Subtraction

The addition of two vectors is performed by adding their corresponding components:

$$\vec{C} = \vec{A} \pm \vec{B}$$

Means the Addition and subtraction of the components side by side:

$$C_i = A_i \pm B_i$$

Example :

Let: $A_i = (1, 2, 3)$ and $B_i = (4, -1, 2)$

The vector sum is: $C_i = A_i + B_i$

Component-wise calculation:

$$C_1 = 1 + 4 = 5$$

$$C_2 = 2 + (-1) = 1$$

$$C_3 = 3 + 2 = 5$$

$$C_i = (5, 1, 5) \text{ or } \vec{C} = 5\vec{e}_1 + \vec{e}_2 + 5\vec{e}_3$$

4.3.2 Scalar Multiplication

Multiplication of a vector by a scalar λ changes its magnitude without altering its direction:

$$\vec{C} = \lambda \vec{A}$$

The components of the resulting vector are:

$$C_i = \lambda A_i$$

If $\lambda > 0$, the vector keeps the same direction; if $\lambda < 0$, its direction is reversed.

4.3.3 Dot Product (Scalar Product)

The dot product of two vectors $\vec{A}$ and $\vec{B}$ is a scalar quantity defined by using Einstein Summation:

$$\vec{A} \cdot \vec{B} = A_i B_i$$

In Euclidean space, it can also be written as:

$$\vec{A} \cdot \vec{B} = \|\vec{A}\| \|\vec{B}\| \cos \theta$$

where θ is the angle between the two vectors.

Example

Consider: $A_i = (2, 1, -1)$ and $B_i = (3, 4, 2)$

Using Einstein summation convention, the dot product is written as:

$$\begin{aligned} A_i B_i &= A_1 B_1 + A_2 B_2 + A_3 B_3 \\ A_i B_i &= (2)(3) + (1)(4) + (-1)(2) = 8 \\ \vec{A} \cdot \vec{B} &= 8 \end{aligned}$$

The dot product is frequently used to determine:

- the angle between two directions,
- vector projections,
- the mechanical work of a force.

4.3.4 Cross Product (Vector Product)

The cross product of two vectors produces a new vector perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$

$$\vec{C} = \vec{A} \times \vec{B}$$

Its magnitude is given by:

$$\|\vec{C}\| = \|\vec{A}\| \|\vec{B}\| \sin \theta$$

Using index notation:

$$C_i = \varepsilon_{ijk} A_j B_k$$

where ε_{ijk} is the Levi-Civita symbol.

Example

Let $\vec{A}$ and $\vec{B}$ have those components $A_i = (1, 2, 3)$; $B_i = (4, 5, 6)$

The cross product is: $\vec{C} = \vec{A} \times \vec{B}$

Using Levi-Civita symbol is $C_i = \varepsilon_{ijk} A_j B_k$

First Component C_1

$$C_1 = \varepsilon_{1jk} A_j B_k$$

The indices j and k vary from 1 to 3. According to the definition of Levi-Civita symbol, whenever two indices are equal, $\varepsilon_{ijk} = 0$. The only nonzero terms are: $\varepsilon_{123} = +1$ and $\varepsilon_{132} = -1$. Therefore:

$$\begin{aligned} C_1 &= \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2 \\ C_1 &= (+1)(2)(6) + (-1)(3)(5) = -3 \end{aligned}$$

Second Component C_2

$$\begin{aligned} C_2 &= \varepsilon_{2jk} A_j B_k \\ C_2 &= \varepsilon_{231} A_3 B_1 + \varepsilon_{213} A_1 B_3 \\ C_2 &= (+1)(3)(4) + (-1)(1)(6) = 6 \end{aligned}$$

Third Component C_3

$$\begin{aligned} C_3 &= \varepsilon_{3jk} A_j B_k \\ C_3 &= \varepsilon_{312} A_1 B_2 + \varepsilon_{321} A_2 B_1 \\ C_3 &= (+1)(1)(5) + (-1)(2)(4) \end{aligned}$$

Combining all components:

$$\vec{C} = (C_1, C_2, C_3) = (-3, 6, -3)$$

The same result can be verified using the determinant form:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

The cross product is commonly used in:

- moment calculations,
- rigid body mechanics,
- rotational mechanics.

4.3.5 Norm of a Vector

The norm of a vector represents its magnitude or length:

$$\|\vec{A}\| = \sqrt{A_i A_i}$$

The vector norm is widely used in mechanics to evaluate the intensity of physical quantities such as displacement, force, and velocity.

4.3.6 Trace of a Tensor

For the tensor:

$$T_{ij} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & 4 \\ 1 & 2 & 6 \end{bmatrix}$$

the trace is defined by:

$$tr(T) = T_{ii}$$

Applying Einstein summation:

$$\begin{aligned} tr(T) &= T_{ii} = T_{11} + T_{22} + T_{33} \\ &= 2 + 5 + 6 = 13 \end{aligned}$$

The trace is an important invariant quantity in elasticity and stress analysis.

4.3.7 Tensor Product

The tensor product of two vectors produces a second-order tensor:

$$T = \vec{A} \otimes \vec{B}$$

The components of the resulting tensor are:

$$T_{ij} = A_i B_j$$

This operation plays a fundamental role in the formulation of stress and strain tensors in elasticity theory.

Example

Consider the two vectors: $\vec{A} = (1, 2, 3)$ and $\vec{B} = (4, -1, 2)$

We want to calculate the tensor product:

$$T = \vec{A} \otimes \vec{B}$$

Apply the Tensor Product Formula

$$T_{ij} = A_i B_j$$

each tensor component is obtained by multiplying one component of $\vec{A}$ by one component of $\vec{B}$.

First Row ($i = 1$)

- $T_{11} = A_1 B_1 = (1)(4) = 4$
- $T_{12} = A_1 B_2 = (1)(-1) = -1$
- $T_{13} = A_1 B_3 = (1)(2) = 2$

Second Row (i=2)

- $T_{21} = A_2 B_1 = (2)(4) = 8$
- $T_{22} = A_2 B_2 = (2)(-1) = -2$
- $T_{23} = A_2 B_3 = (2)(2) = 4$

Third Row (i=3)

- $T_{31} = A_3 B_1 = (3)(4) = 12$
- $T_{33} = A_3 B_3 = (3)(2) = 6$
- $T_{33} = A_3 B_3 = (3)(2) = 6$

Combining all components:

$$T = \vec{A} \otimes \vec{B} = \begin{bmatrix} 4 & -1 & 2 \\ 8 & -2 & 4 \\ 12 & -3 & 6 \end{bmatrix}$$

5. Conclusion

This chapter introduced the fundamental mathematical concepts used in Elasticity and continuum mechanics. Scalars, vectors, and tensors were presented as essential tools for describing physical quantities such as displacement, force, stress, and strain. The principles of index notation and Einstein summation convention were discussed to provide a compact and efficient formulation of vector and tensor equations. Basic vector and tensor operations, including dot products, cross products, and tensor products, were also examined in detail. Special mathematical tools such as the Kronecker delta and the Levi-Civita symbol were introduced for advanced tensor manipulations. These mathematical foundations constitute the basis for the formulation of equilibrium equations, constitutive laws, and deformation analysis developed in the following chapters of elasticity theory.

Exercises

Exercise 1

Two points A and B have the coordinates: A(2, 3, -3) ; B(5, 7, -3)

1. Determine the magnitude, the direction (direction cosines), and the sense of the vector AB.
2. Determine the components of the vector V perpendicular to AB and belonging to the plane (O, x, y).

Exercise 2

Let the vectors be: $\vec{V}_1 = 2\vec{i} - \vec{j} + 5\vec{k}$, $\vec{V}_2 = -3\vec{i} + 1.5\vec{j} - 7.5\vec{k}$, $\vec{V}_3 = -5\vec{i} + 4\vec{j} + \vec{k}$

1. Calculate the dot products: $\vec{V}_1 \cdot \vec{V}_2$; $\vec{V}_1 \cdot \vec{V}_1$; $\vec{V}_2 \cdot \vec{V}_2$
2. Calculate the cross products: $\vec{V}_1 \times \vec{V}_2$; $\vec{V}_2 \times \vec{V}_3$; $\vec{V}_3 \times \vec{V}_2$
3. Calculate: $\vec{V}_1 \cdot (\vec{V}_2 \times \vec{V}_3)$, $\vec{V}_1 \times (\vec{V}_2 \times \vec{V}_3)$

Exercise 3

In an orthonormal coordinate system, the vectors are given by: $\vec{V}_1 = 3\vec{i} + 5\vec{k}$, $\vec{V}_2 = 5\vec{i} + y\vec{j} + z\vec{k}$, $\vec{V}_3 = 3\vec{i} + 2y\vec{j} - 2\vec{k}$, $\vec{V}_4 = x\vec{i} + 2\vec{j} + 10\vec{k}$

1. Determine y and z so that vectors $\vec{V}_1$ and $\vec{V}_2$ are collinear.
2. Determine x so that vectors $\vec{V}_3$ and $\vec{V}_4$ are perpendicular.

Exercise 4

In a coordinate system $(O, \vec{x}, \vec{y}, \vec{z})$, consider the three vectors:

$$\vec{U}(0,1,1), \quad \vec{V}(-1,0,1), \quad \vec{W}(0,-1,1)$$

Determine the tensor $\bar{T}$ such that: $\bar{T} \cdot \vec{U} = \vec{A}$; $\bar{T} \cdot \vec{V} = \vec{B}$; $\bar{T} \cdot \vec{W} = \vec{C}$
where:

$$\vec{A} = \begin{pmatrix} 2b \\ a+b-c \\ a+b+c \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} b+c-a \\ -2c \\ c+a-b \end{pmatrix}, \quad \vec{C} = \begin{pmatrix} 2c \\ b-c-a \\ a-b-c \end{pmatrix}$$

1. Determine the symmetric tensor $\bar{D}$ and the antisymmetric tensor $\bar{G}$ components of tensor $\bar{T}$.
2. How should the values (a,b,c) be chosen so that tensor $\bar{T}$ is symmetric?

Exercise 5

Let $(\vec{i}, \vec{j})$ be a basis of a vector space E_2 , and let two vectors of E_2 be:

$$\vec{x} = 2\vec{i} + 4\vec{j} \quad \vec{y} = 5\vec{i} + 3\vec{j}$$

1. Determine the expression of the tensor product: $\vec{x} \otimes \vec{y}$
2. The following tensor: $\bar{U} = 11\vec{i} \otimes \vec{i} + 8\vec{i} \otimes \vec{j} + 20\vec{j} \otimes \vec{i} + 12\vec{j} \otimes \vec{j}$. Is it a tensor product in E_2 ?
3. Show that tensor $\bar{U}$ is the sum of the tensor product $\vec{x} \otimes \vec{y}$ and another tensor $\bar{W}$ to be determined.

Solutions

Solution to exercise 1

1) Vector $\vec{AB}$: $\vec{AB} = (5-2, 7-3, -3-(-3)) = (3, 4, 0)$

Magnitude: $|\vec{AB}| = \sqrt{x^2 + y^2 + z^2} = 5$

Direction cosines: $(\cos \alpha = \frac{3}{5}, \cos \beta = \frac{4}{5}, \cos \gamma = \frac{0}{5}) = (\frac{3}{5}, \frac{4}{5}, 0)$

2) Vector perpendicular to $\vec{AB}$ in plane (Oxy)

A vector in plane (Oxy) has form: $\vec{V} = (x, y, 0)$

Orthogonality condition: $\vec{V} \cdot \vec{AB} = 0$

$$(x, y, 0) \cdot (3, 4, 0) = 3x + 4y = 0$$

Take: $x = 4$ Then: $3(4) + 4y = 0$ so $y = -3$

One possible vector is: $\vec{V} = (4, -3, 0)$

Solution to exercise 2

1) Dot products

$$\vec{A} \cdot \vec{B} = x_1x_2 + y_1y_2 + z_1z_2 = 2(-3) + (-1)(1.5) + 5(-7.5) = -45$$

$$\vec{V}_1 \cdot \vec{V}_1 = 2^2 + (-1)^2 + 5^2 = 30$$

$$\vec{V}_2 \cdot \vec{V}_2 = (-3)^2 + (1.5)^2 + (-7.5)^2 = 67.5$$

2) Cross products

$$\begin{aligned}
 \bullet \quad \vec{V}_1 \times \vec{V}_2 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 5 \\ -3 & 1.5 & -7.5 \end{vmatrix} \\
 &= \vec{i}[(-1)(-7.5) - 5(1.5)] - \vec{j}[2(-7.5) - 5(-3)] + \vec{k}[2(1.5) - (-1)(-3)] \\
 &= \vec{i}(7.5 - 7.5) - \vec{j}(-15 + 15) + \vec{k}(3 - 3) = (0,0,0)
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad \vec{V}_2 \times \vec{V}_3 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1.5 & -7.5 \\ -5 & 4 & 1 \end{vmatrix} \\
 &= \vec{i}(1.5 + 30) - \vec{j}(-3 - 37.5) + \vec{k}(-12 + 7.5) = (31.5, \quad 40.5, \quad -4.5)
 \end{aligned}$$

$$\bullet \quad \vec{V}_3 \times \vec{V}_2$$

Using the property: $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$

$$\vec{V}_3 \times \vec{V}_2 = -(31.5, 40.5, -4.5)$$

3) Triple products

- Scalar triple product

$$\vec{V}_1 \cdot (\vec{V}_2 \times \vec{V}_3) = (2, -1, 5) \cdot (31.5, 40.5, -4.5) = 2(31.5) + (-1)(40.5) + 5(-4.5) = 0$$

- Vector triple product

$$\text{Using this } \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\vec{V}_1 \times (\vec{V}_2 \times \vec{V}_3) = -9\vec{V}_2 + 45\vec{V}_3$$

Solution to exercise 3

1. Determine y and z so that $\vec{V}_1$ and $\vec{V}_2$ are collinear

Write vectors in component form: $\vec{V}_1 = (3, 0, 5)$, $\vec{V}_2 = (5, y, z)$

Two vectors are collinear if: $\vec{V}_2 = \lambda \vec{V}_1$

$$(5, y, z) = \lambda(3, 0, 5)$$

So $5 = 3\lambda$, $y = 0$, $z = 5\lambda$

$$\lambda = \frac{5}{3}, \quad y = 0; \quad z = 5\left(\frac{5}{3}\right) = \frac{25}{3}$$

2) Determine xxx so that $\vec{V}_3$ and $\vec{V}_4$ are perpendicular

First substitute $y=0$: $\vec{V}_3 = 3\vec{i} - 2\vec{k}$

Component form: $\vec{V}_3 = (3, 0, -2)$, $\vec{V}_4 = (x, 2, 10)$

$$\begin{aligned}
 \text{Two vectors are perpendicular if: } \vec{V}_3 \cdot \vec{V}_4 &= (3, 0, -2) \cdot (x, 2, 10) = 3x + 0(2) + (-2)(10) = 0 \\
 \Rightarrow 3x &= 20 \Rightarrow x = \frac{20}{3}
 \end{aligned}$$

Solution to exercise 4

1) Determine the tensor $\bar{\bar{T}}$

$$\bar{T} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

Using $\bar{T}\vec{U} = \vec{A}$, $\vec{U} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\bar{T}\vec{U} = \begin{pmatrix} t_{12} + t_{13} \\ t_{22} + t_{23} \\ t_{32} + t_{33} \end{pmatrix} = \begin{pmatrix} 2b \\ a + b - c \\ a + b + c \end{pmatrix}$$

$$t_{12} + t_{13} = 2b$$

$$t_{12} + t_{13} = 2b$$

$$t_{32} + t_{33} = a + b + c$$

Using $\bar{T}\vec{V} = \vec{B}$, $\vec{V} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\bar{T}\vec{V} = \begin{pmatrix} -t_{11} + t_{13} \\ -t_{21} + t_{23} \\ -t_{31} + t_{33} \end{pmatrix} = \begin{pmatrix} b + c - a \\ -2c \\ c + a - b \end{pmatrix}$$

$$-t_{11} + t_{13} = b + c - a$$

$$-t_{11} + t_{13} = b + c - a$$

$$-t_{31} + t_{33} = c + a - b$$

Using $\bar{T}\vec{W} = \vec{C}$, $\vec{W} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$$\bar{T}\vec{W} = \begin{pmatrix} -t_{12} + t_{13} \\ -t_{22} + t_{23} \\ -t_{32} + t_{33} \end{pmatrix} = \begin{pmatrix} 2c \\ b - c - a \\ a - b - c \end{pmatrix}$$

After determine the coefficients we have

$$\bar{T} = \begin{pmatrix} a & b - c & b + c \\ b + c & a & b - c \\ b - c & b + c & a \end{pmatrix}$$

2) Determine the symmetric tensor $\bar{D}$

$$\bar{D} = \frac{1}{2}(\bar{T} + \bar{T}^T)$$

$$\bar{T}^T = \begin{pmatrix} a & b + c & b - c \\ b - c & a & b + c \\ b + c & b - c & a \end{pmatrix}, \quad \bar{T} + \bar{T}^T = \begin{pmatrix} 2a & 2b & 2b \\ 2b & 2a & 2b \\ 2b & 2b & 2a \end{pmatrix}$$

$$\bar{D} = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$$

3) Determine the antisymmetric tensor $\bar{G}$

$$\bar{G} = \frac{1}{2}(\bar{T} - \bar{T}^T) = \begin{pmatrix} 0 & -c & c \\ c & 0 & -c \\ -c & c & 0 \end{pmatrix}$$

4) Condition for to be symmetric $\bar{T}$

A tensor is symmetric if: $\bar{T} = \bar{T}^T$

Thus antisymmetric part must vanish: $\bar{G} = 0$ means $c = 0$

Soluion to exercise 5

1) Determine the tensor product $\vec{x} \otimes \vec{y} = (2\vec{i} + 4\vec{j}) \otimes (5\vec{i} + 3\vec{j})$

Use distributivity:
$$= 2(\overset{\rightarrow}{5i} \otimes \vec{i} + \overset{\rightarrow}{3i} \otimes \vec{j}) + 4(\overset{\rightarrow}{5j} \otimes \vec{i} + \overset{\rightarrow}{3j} \otimes \vec{j})$$
$$\vec{x} \otimes \vec{y} = 10\vec{i} \otimes \vec{i} + 6\vec{i} \otimes \vec{j} + 20\vec{j} \otimes \vec{i} + 12\vec{j} \otimes \vec{j}$$

2) Is $\bar{\bar{U}}$ a tensor product in E_2 ?

Write $\bar{\bar{U}}$ in matrix form:
$$\bar{\bar{U}} = \begin{pmatrix} 11 & 8 \\ 20 & 12 \end{pmatrix}$$

A tensor is a simple tensor product if: $\det(\bar{\bar{U}}) = 0$

$$\det(\bar{\bar{U}}) = 11(12) - 8(20) = -28, \quad \bar{\bar{U}} \text{ is not a simple tensor product.}$$

$\bar{\bar{U}}$ is not a tensor product in E_2

Chapter 2: Theory of Stress State

1. Introduction
2. Stress tensor
3. Differential equations of equilibrium
4. Particular Stress Tensors
5. Geometrical representation (Mohr's circle)
6. Conclusion

1. Introduction

Using a cutting surface S , let us isolate a part (1) of a solid body Ω as shown in Figure 2.1. On the cut surface, we consider a small area element dA , called a surface element or facet, with an outward unit normal vector $\vec{n}$ (taken as positive by convention).

The interaction between the material of fragment (1) and fragment (2) appears in the form of internal forces acting across all the surface elements dA that make up the cutting surface. These internal forces are responsible for the development of stresses inside the material.

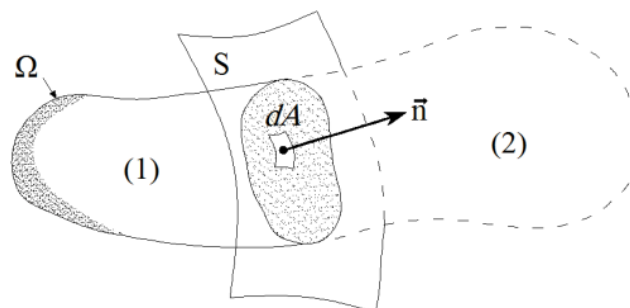


Figure 2.1 Cutting surface S with element dA and normal vector n .

The study of the stress state makes it possible to understand how forces are distributed within a solid body and how the material reacts under external loading. This chapter introduces the stress tensor, equilibrium equations, stresses acting on different planes, principal stresses, and their graphical representation using Mohr's circle.

2. Stress tensor

The concept of the stress tensor is fundamental in continuum mechanics and elasticity. When a solid body is subjected to external forces, internal forces develop inside the material in order to maintain equilibrium. These internal forces are distributed over imaginary surfaces within the body and are represented mathematically by stresses.

Since stresses may act in different directions on different planes passing through the same point, a simple scalar or vector description is not sufficient. For this reason, the state of stress at a point is represented by a second-order tensor called the stress tensor. This tensor completely describes the distribution of internal forces inside a deformable body.

The stress tensor allows the determination of:

- normal stresses acting perpendicular to a surface,
- shear stresses acting tangentially to a surface,
- stresses acting on arbitrary planes,
- principal stresses and their directions.

To introduce the stress tensor progressively, this section is divided into the following subsections.

2.1 Euler–Cauchy Principle

The Euler–Cauchy principle states that the internal forces acting on an infinitesimal surface element inside a continuous medium can be represented by a force vector per unit area called the stress vector or traction vector.

Consider an infinitesimal surface element dS with outward unit normal vector $\vec{n}$. The material located on one side of the surface exerts an internal force $d\vec{F}$ on the material located on the other side. The stress vector is defined by:

$$\vec{T}^{(n)} = \lim_{dA \rightarrow 0} \frac{d\vec{F}}{dS}$$

The vector $\vec{T}^{(n)}$ depends both on the considered point and on the orientation of the surface defined by $\vec{n}$.

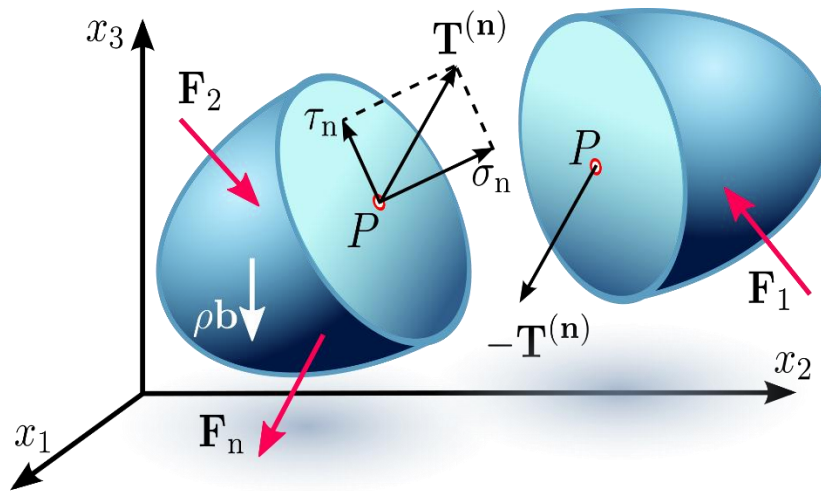


Figure 2.2 Euler–Cauchy stress principle.

This principle forms the basis for the mathematical formulation of stresses in continuum mechanics.

2.2 Stress Components

Consider a material parallelepiped constructed around the origin O of a Cartesian coordinate system (x_i) (Figure 3).

On the three positive faces normal to the coordinate axes act the stress vectors $\vec{f}_1$, $\vec{f}_2$, and $\vec{f}_3$ (Figure 3a). Each stress vector has three components parallel to the coordinate axes:

- one normal component σ_{ii} (no summation),
- two tangential or shear components σ_{ij} with ($i \neq j$).

The first index indicates the face on which the stress component acts, while the second index indicates the direction parallel to the corresponding axis. A stress component is considered positive when it acts in the positive direction of the axis on a positive face (Figure 3b). On the negative face, the positive direction is reversed according to the principle of action and reaction.

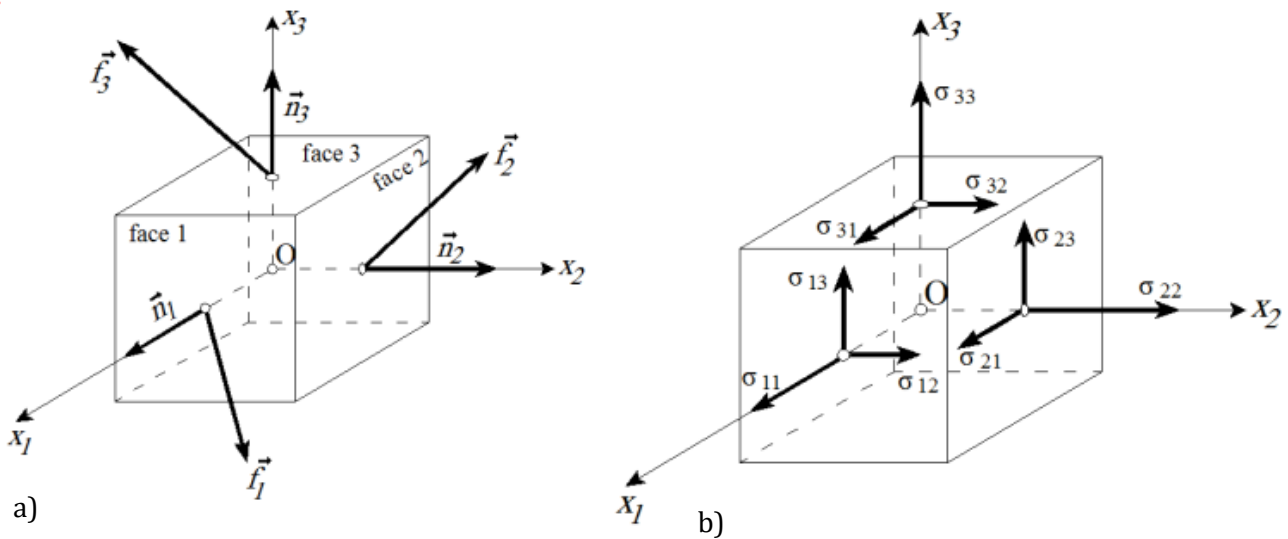


Figure 2.3 Sign convention and notation: (a) Stress vectors acting on three surface elements, (b) Stress components σ_{ij} .

For clarity in the figure, the faces are represented with finite dimensions. In reality, these surface elements must be imagined as infinitesimally small and all centered at the same point O , which is the origin of the coordinate system. Therefore, the quantities σ_{ij} represent the components of three stress vectors acting at the same point on three mutually orthogonal planes.

The nine stress components are conveniently arranged in the following matrix form, called the **stress tensor**. The first representation uses index notation, while the second follows the notation commonly used in engineering mechanics. Each row corresponds to one stress vector:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Remark

In index notation, the symbol σ is used for both normal and shear stresses. However, in engineering notation, shear stresses are usually denoted by the symbol τ , while the symbol σ is reserved for normal stresses.

2.3 Normal and Tangential Stresses

The stress vector acting on a surface can be separated into (Figure 2.4):

- a component perpendicular to the surface called the normal stress,
- a component parallel to the surface called the tangential stress or shear stress.

The normal stress is given by:

$$\sigma_n = \vec{T}^{(n)} \cdot \vec{n}$$

It represents tensile or compressive effects inside the material.

The tangential stress vector is:

$$\vec{\tau} = \vec{T}^{(n)} - \sigma_n \vec{n}$$

It represents sliding effects between adjacent material layers.

Normal stresses are generally denoted by σ , while shear stresses are denoted by τ .

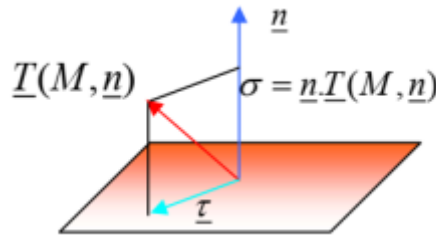


Figure 2.4 – Decomposition of the stress vector acting at point M on the surface element with normal vector $\underline{n}$.

In Figure 2.5, Considering in small parallelepiped surrounding point P .

Let $\vec{T}_1, \vec{T}_2$, and $\vec{T}_3$, be the stress vectors associated with the surfaces whose outward normal unite vectors are $\vec{e}_1, \vec{e}_2$, and $\vec{e}_3$, , respectively. Each of these stress vectors can be decomposed along the three coordinate axes x_1, x_2 , and x_3 (Figure 4). The nine components of the three stress vectors can therefore be written in the following form:

$$T_1 = \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix}, T_2 = \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{pmatrix}, T_3 = \begin{pmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix}$$

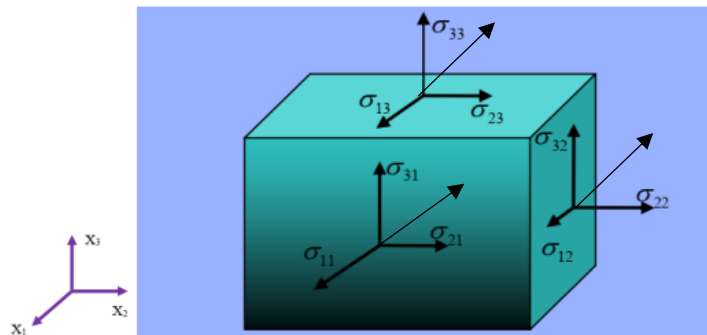


Figure 2.5 – Normal and tangential Components of the stress vectors in 3d case.

As an example lets compute the scalar product:

$$\vec{T}_2 \cdot \vec{e}_1 = |\vec{T}_2| |\vec{e}_1| \cos(\vec{T}_2, \vec{e}_1)$$

Since : $\cos(T_2, e_1) = \frac{\sigma_{21}}{|\vec{T}_2|}$

we obtain: $\sigma_{21} = |\vec{T}_2| \cos(T_2, e_1)$

Because: $|\vec{T}_2| = T_2$ and $|\vec{e}_1| = 1$

Then

$$\sigma_{21} = T_2 \cos(\vec{T}_2, \vec{e}_1) = \vec{T}_2 \cdot \vec{e}_1$$

Similarly:

$$\begin{aligned} \sigma_{22} &= \vec{T}_2 \cdot \vec{e}_2; & \sigma_{23} &= \vec{T}_2 \cdot \vec{e}_3 \\ \sigma_{11} &= \vec{T}_1 \cdot \vec{e}_1; & \sigma_{12} &= \vec{T}_1 \cdot \vec{e}_2 \\ \sigma_{31} &= \vec{T}_3 \cdot \vec{e}_1; & \sigma_{33} &= \vec{T}_3 \cdot \vec{e}_3 \end{aligned}$$

From these relations, we deduce the general expression:

$$\sigma_{ij} = T_i \cdot e_j$$

The components σ_{11} , σ_{22} and σ_{33} represent the normal components of the stress tensor σ_{ij} whereas the off-diagonal components such as σ_{12} , σ_{23} , and σ_{13} represent the shear stresses. The first index indicates the direction in which the stress component acts, while the second index indicates the normal direction of the surface element.

2.4 Cauchy's Theorem

To establish Cauchy's theorem, let us consider an infinitesimally small tetrahedron MABC surrounding a point M inside the solid body. The tetrahedron is aligned with the Cartesian axes x_1 , x_2 , and x_3 (Figure 2.6).

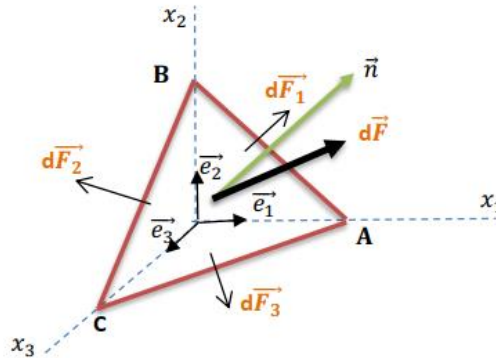


Figure 2.6 Tetrahedron under the action of surface forces.

Let $\vec{n}$ be the outward unit normal vector to the plane (ABC), whose components are (n_1, n_2, n_3) . The area of the triangular face ABC is denoted by dS .

A surface force $d\vec{F}$ acts on the inclined face ABC, while the forces $d\vec{F}_1$, $d\vec{F}_2$, and $d\vec{F}_3$ act on the three faces perpendicular to the coordinate axes. These forces represent the internal stresses acting on the infinitesimal tetrahedron.

Equilibrium of Forces

For the tetrahedron to remain in equilibrium, the sum of all surface forces must be zero:

$$d\vec{F} + d\vec{F}_1 + d\vec{F}_2 + d\vec{F}_3 = 0$$

Using the traction vectors, this relation becomes:

$$dS \vec{T}(M, \vec{n}) + dS_1 \vec{T}(M, -\vec{e}_1) + dS_2 \vec{T}(M, -\vec{e}_2) + dS_3 \vec{T}(M, -\vec{e}_3) = 0$$

The projected areas of the coordinate planes satisfy:

$$dS_1 = n_1 dS, dS_2 = n_2 dS, dS_3 = n_3 dS$$

Moreover, according to the action-reaction principle:

$$\vec{T}(M, -\vec{n}) = -\vec{T}(M, \vec{n})$$

Substituting these relations into the equilibrium equation gives:

$$dS \vec{T}(M, \vec{n}) + n_1 dS \vec{T}(M, -\vec{e}_1) + n_2 dS \vec{T}(M, -\vec{e}_2) + n_3 dS \vec{T}(M, -\vec{e}_3) = 0$$

This equation shows that the traction vector acting on any plane with normal vector $\vec{n}$ can be expressed as a linear combination of the traction vectors acting on the coordinate planes.

Matrix Form

In matrix notation, the relation may be written as:

$$\{\vec{T}(M, \vec{n})\} = \begin{bmatrix} \vec{T}(M, \vec{e}_1) & \vec{T}(M, \vec{e}_2) & \vec{T}(M, \vec{e}_3) \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

Therefore, at every point M, there exists a second-order tensor called the **Cauchy stress tensor**, which defines a linear mapping between the normal vector $\vec{n}$ and the traction vector $\vec{T}(M, \vec{n})$:

$$\vec{T}(M, \vec{n}) = \sigma(M) \cdot \vec{n}$$

where the Cauchy stress tensor is written as:

$$\sigma(M) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

In index notation, the traction components are expressed as:

$$T_i(M, \vec{n}) = \sigma_{ij} n_j$$

The Cauchy stress tensor $\sigma(M)$ is symmetric ($\sigma_{ij} = \sigma_{ji}$), This symmetry results from the rotational equilibrium of the infinitesimal element.

2.5 Principal Stresses and principal directions

The state of stress at a point inside a material depends on the orientation of the considered plane. Among all possible planes passing through a point, there exist particular planes on which the shear stresses vanish completely (Figure 2.7) .

$$\vec{T}(M, \vec{n}) = \sigma(M) \cdot \vec{n} = \sigma_n \vec{n}, \quad \tau_n = 0$$

The normal stresses acting on these planes are called principal stresses ($\sigma_I, \sigma_{II}, \sigma_{III}$), and the corresponding directions are known as the principal directions ($\vec{e}_I, \vec{e}_{II}, \vec{e}_{III}$).

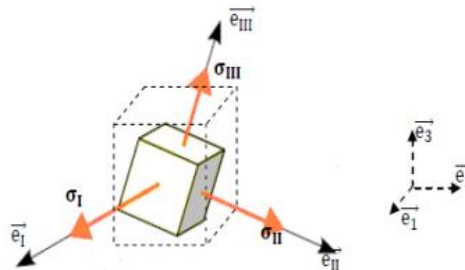


Figure 2.7 Principal stresses and principal directions.

The quantity σ_n is then an eigenvalue of the stress tensor $\sigma(M)$, while the vector $\vec{n}$ is the associated eigenvector. Since the stress tensor σ is symmetric, it can always be diagonalized in a suitable coordinate system:

$$\sigma(M) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

For a general three-dimensional stress state, the traction vector acting on a plane of normal vector $\vec{n}$ is given by:

$$\vec{T} = \sigma \cdot \vec{n}$$

On a principal plane, the traction vector becomes purely normal to the plane, meaning that it is parallel to $\vec{n}$. Therefore:

$$\vec{T} = \lambda \vec{n}$$

where λ represents a principal stress.

$$\sigma \cdot \vec{n} = \lambda \vec{n}$$

or equivalently:

$$(\sigma - \lambda I)\vec{n} = 0$$

For a nontrivial solution ($\vec{n} \neq 0$), the determinant must vanish:

$$\det(\sigma - \lambda I) = 0$$

This equation is called the characteristic equation of the stress tensor. Solving it provides the three principal stresses:

$$\sigma_I > \sigma_{II} > \sigma_{III}$$

The associated eigenvectors define the principal directions $\vec{e}_I, \vec{e}_{II}, \vec{e}_{III}$.

The determination of principal stresses is very important in solid mechanics because they represent the extreme values of normal stress experienced by the material and are frequently used in failure criteria and structural analysis.

2.6 Stress Invariants

The coefficients of the characteristic equation are independent of the coordinate system. These quantities are called the stress invariants.

For the stress tensor σ , the three invariants are:

2.6.1 First Invariant

The first invariant represents the trace of the stress tensor and is related to the volumetric effect of stresses.

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

2.6.2 Second Invariant

This invariant combines both normal and shear stress effects.

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

2.6.3 Third Invariant

The third invariant corresponds to the determinant of the stress tensor.

$$I_3 = \det(\sigma)$$

Using these invariants, the characteristic equation may be written as:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

The roots of this cubic equation are the three principal stresses. Because the invariants do not depend on the orientation of the coordinate system, they provide an intrinsic description of the stress state at a point.

3. Differential equations of equilibrium

Consider an infinitesimal parallelepiped particle of dimensions $(\Delta x_1, \Delta x_2, \Delta x_3)$ (Figure 2.8). To establish the differential equations of equilibrium, we apply the law of conservation of linear momentum along the axis x_1 .

$$\int_V \rho \vec{g} dV + \int_S \vec{\tau} dS = \int_V \rho \vec{\gamma} dV$$

Three types of forces act on the particle:

- Body forces due to gravity: $\int_V \rho \vec{g} dV$
- Surface forces generated by stresses acting on the faces of the element: $\int_S \vec{\tau} dS$
- Inertial forces associated with acceleration: $\int_V \rho \vec{\gamma} dV$

where: ρ is the mass density, $\vec{g}$ is the body force per unit mass, $\vec{\gamma}$ is the acceleration vector.

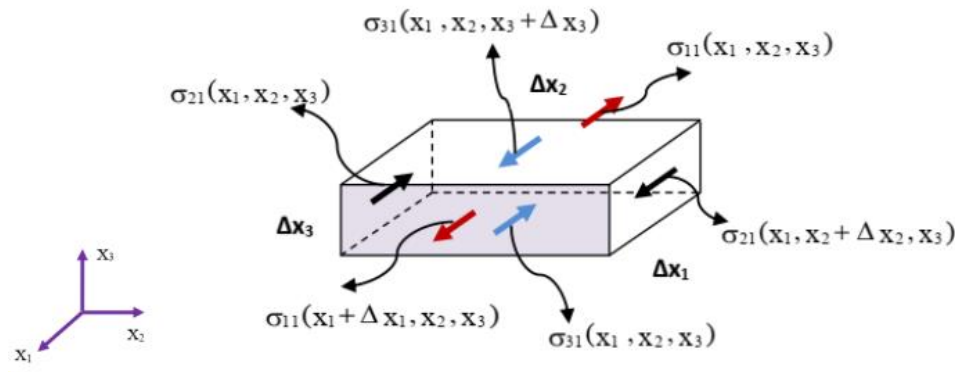


Figure 2.8 infinitesimal parallelepiped particle under external load.

3.1 Surface Forces

The resultant surface force along the direction x_1 is obtained by summing the stresses acting on the opposite faces of the infinitesimal element.

Thus:

$$\begin{aligned} \int_S \vec{\tau} dS &= \iint [\sigma_{11}(x_1 + \Delta x_1, x_2, x_3) - \sigma_{11}(x_1, x_2, x_3)] dx_2 dx_3 \\ &+ \iint [\sigma_{21}(x_1, x_2 + \Delta x_2, x_3) - \sigma_{21}(x_1, x_2, x_3)] dx_1 dx_3 \\ &+ \iint [\sigma_{31}(x_1, x_2, x_3 + \Delta x_3) - \sigma_{31}(x_1, x_2, x_3)] dx_1 dx_2 \end{aligned}$$

To simplify these expressions, we use a first-order Taylor series expansion.

For example:

$$\sigma_{11}(x_1 + \Delta x_1, x_2, x_3) - \sigma_{11}(x_1, x_2, x_3) = \frac{\partial \sigma_{11}}{\partial x_1} \Delta x_1$$

Similarly:

$$\sigma_{21}(x_1, x_2 + \Delta x_2, x_3) - \sigma_{21}(x_1, x_2, x_3) = \frac{\partial \sigma_{21}}{\partial x_2} \Delta x_2$$

$$\sigma_{31}(x_1, x_2, x_3 + \Delta x_3) - \sigma_{31}(x_1, x_2, x_3) = \frac{\partial \sigma_{31}}{\partial x_3} \Delta x_3$$

Substituting these results into the surface force expression gives:

$$\int_S \vec{\tau} dS = \iiint_V \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} \right) dV$$

The volume of the infinitesimal particle is:

$$dV = dx_1 dx_2 dx_3$$

Since the element dimensions are infinitesimally small, we may use the approximation:

$$dx_i \approx \Delta x_i$$

3.2 Body Forces and Inertial Forces

The body forces acting on the particle along the direction x_1 are :

$$\iiint_V \rho g_{x_1} dV$$

while the inertial forces are:

$$\iiint_V \rho \gamma_{x_1} dV$$

Applying Newton's second law along the axis x_1 , we obtain:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho g_{x_1} = \rho \gamma_{x_1} = \rho \frac{d^2 x_1}{dt^2}$$

In the same way, equilibrium along the axes x_2 and x_3 leads to :

$$\begin{aligned} \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho g_{x_2} &= \rho \frac{d^2 x_2}{dt^2} \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho g_{x_3} &= \rho \frac{d^2 x_3}{dt^2} \end{aligned}$$

Final Form of the Equilibrium Equations

By introducing the notation: $f_i = \rho g_{x_i}$

the differential equations of equilibrium become:

$$\begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = \rho \frac{d^2 x_1}{dt^2} \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = \rho \frac{d^2 x_2}{dt^2} \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = \rho \frac{d^2 x_3}{dt^2} \end{cases}$$

Index Notation

Using Einstein's summation convention, the equilibrium equations can be written in compact tensor form:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho \frac{d^2 x_i}{dt^2}$$

These equations are called the differential equations of equilibrium or the equations of motion of a continuous medium. They express the balance between internal stresses, external body forces, and inertial effects inside the material.

4. Particular Stress Tensors

Certain stress states occur frequently in mechanics and engineering applications. These special forms of the stress tensor simplify the analysis of structures and materials under loading. The most important cases are presented below.

4.1 Uniaxial Stress State (Simple Tension or Compression)

The stress state at a point M is said to be uniaxial (Figure 2.9) when only one normal stress component is nonzero. In this case, the stress tensor is written as:

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding traction vector becomes:

$$\vec{T}(M, \vec{n}) = \sigma \cdot \vec{e}_1$$

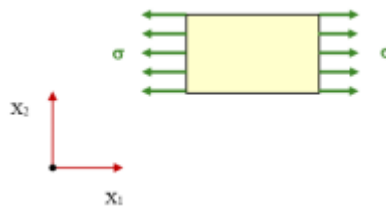


Figure 2.9 Uniaxial stress state.

In this stress state, the material is subjected only to a normal stress along the direction x_1

- If $\sigma > 0$, the material is under simple tension (traction).
- If $\sigma < 0$, the material is under simple compression.

This type of stress state is commonly encountered in tensile and compression tests on bars and rods.

4.2 Plane Stress State

At a point M, the stress state is said to be a plane stress state (Figure 2.10) when all stress components perpendicular to one direction vanish. The stress tensor then takes the form:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In this case, all stresses evolve only in the plane (x_1, x_2) , while stresses involving the direction x_3 are neglected. This approximation is widely used in the analysis of thin plates and shells subjected to in-plane loading.

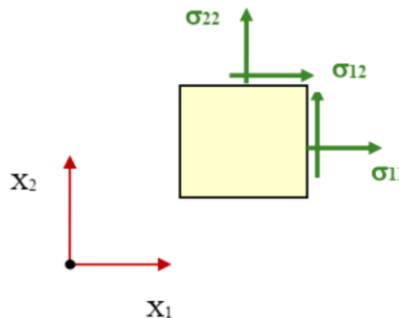


Figure 2.10 Plane stress state.

If, in addition, the normal stress $\sigma_{33} \neq 0$, the stress tensor becomes:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

This situation is referred to as a pseudo-plane stress state.

4.3 Isotropic Stress State

The stress state at a point M is called isotropic or hydrostatic (Figure 13) when the traction vector is always normal to the considered surface, regardless of its orientation:

$$\vec{T}(M, \vec{n}) = \sigma \vec{n}$$

In this case, the three principal stresses are equal, and the stress tensor has the same form in every coordinate system:

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$

- If $\sigma > 0$, the material is subjected to hydrostatic tension.
- If $\sigma < 0$, the material is subjected to hydrostatic compression.

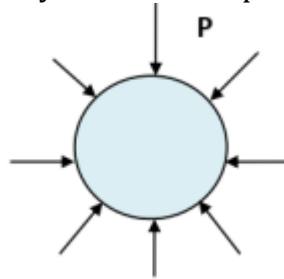


Figure 2.11 Isotropic stress state.

This type of stress state is commonly found in fluids at rest and in materials subjected to uniform pressure.

4.4 Simple Shear Stress State

The stress state at point M is called a simple shear state (Figure 2.12) with respect to the directions x_1 and x_2 when the stress tensor reduces to:

$$\sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

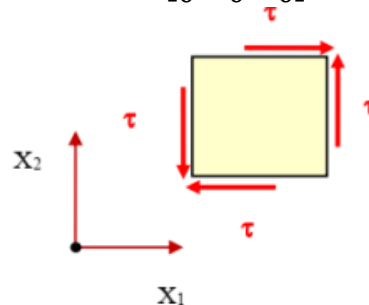


Figure 2.12 – Simple shear stress state.

In this configuration:

- only shear stresses are present,
- normal stresses vanish,
- the material experiences sliding deformation between adjacent layers.

Simple shear is encountered in torsion problems, riveted joints, and adhesive layers.

5. Geometrical representation (Mohr's circle)

5.1 Principle of Mohr's Representation

Mohr's representation is a graphical method used to describe the stress state at a point inside a material. It allows the determination of:

- normal stresses,
- shear stresses,
- principal stresses,
- maximum shear stresses,
- and the orientation of critical planes.

The traction vector acting on a surface of normal vector $\vec{n}$ can be decomposed into (Figure 2.12):

- a normal component σ
- and a tangential (shear) component τ .

Thus:

$$\vec{T}(M, \vec{n}) = \sigma \vec{n} + \tau \vec{t}$$

where:

- $\vec{n}$ is the unit normal vector to the surface,
- $\vec{t}$ is the unit tangent vector to the surface,
- σ the normal stress,
- τ is the shear stress.

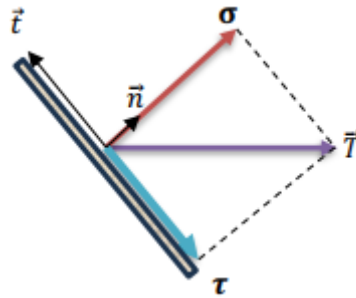


Figure 2.12 Traction vector acting on a surface.

We consider the plane defined by $\vec{n}$ and $\vec{t}$, and orient it such that $(\vec{n}, \vec{t}) = \frac{\pi}{2}$. An auxiliary plane is then introduced, defined by the vectors σ and τ , with $(O\sigma, O\tau) = \frac{\pi}{2}$, in which the components σ and τ are associated with each vector $\vec{T}(M)$ [4 malika].

5.2 Mohr Circle from Principal Stress

Consider a plane stress state where the principal σ_I stresses σ_{II} and are known (Figure 2.13). We seek the stress components σ and τ acting on a plane whose normal makes an angle θ with the principal direction x_I .

The stress tensor in the rotated coordinate system is obtained through a rotation transformation:

$$[\sigma] = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \sigma_I & 0 \\ 0 & \sigma_{II} \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

The resulting stress components are:

$$\begin{aligned} \sigma &= \sigma_I \cos^2 \theta + \sigma_{II} \sin^2 \theta \\ \tau &= (\sigma_I - \sigma_{II}) \cos \theta \sin \theta \end{aligned}$$

Using the trigonometric identities:

$$\cos^2 \theta = \frac{1+\cos 2\theta}{2}; \quad \sin^2 \theta = \frac{1-\cos 2\theta}{2}; \quad \sin 2\theta = 2\sin \theta \cos \theta$$

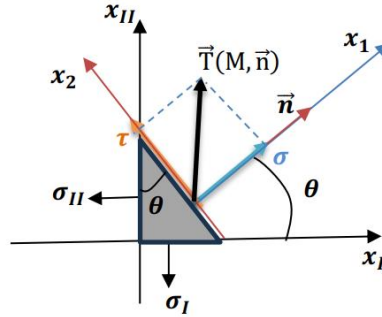


Figure 2.13 Determination of σ and τ from known principal stresses.

The equations become:

$$\sigma = \frac{\sigma_I + \sigma_{II}}{2} + \frac{\sigma_I - \sigma_{II}}{2} \cos 2\theta$$

$$\tau = -\frac{\sigma_I - \sigma_{II}}{2} \sin 2\theta$$

Let: $a = \frac{\sigma_I + \sigma_{II}}{2}$; $r = \frac{\sigma_I - \sigma_{II}}{2}$

Then the stress become:

$$\sigma = a + r \cos 2\theta$$

$$\tau = -r \sin 2\theta$$

Squaring and adding the two equations gives:

$$(\sigma - a)^2 + \tau^2 = r^2$$

This is the equation of a circle in the (σ, τ) plane with a center $(a, 0)$ and a radius r . This circle is called the **Mohr Circle** (Figure 2.14).

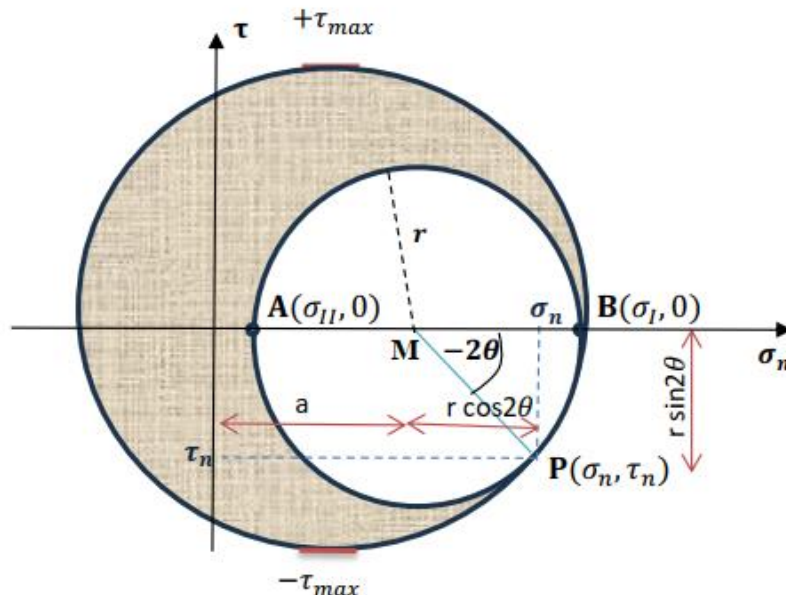


Figure 2.14 Description of Mohr's Circle from the principal stresses..

Important Properties of Mohr's Circle

The Mohr Circle provides a simple graphical interpretation of the stress state:

- Each point on the circle represents the stress state on a particular plane.
- The coordinates of a point correspond to: normal stress and shear stress

- Two perpendicular planes correspond to two diametrically opposite points on the circle.
- The intersections of the circle with the σ -axis give the principal stresses where $\tau=0$.
- The maximum shear stress is equal to the radius of the circle:

$$\tau_{max} = \pm \frac{\sigma_I - \sigma_{II}}{2}$$

5.3 Tri-Circle of Mohr

In a three-dimensional stress state, the stress tensor possesses three principal stresses:

$$\sigma_I > \sigma_{II} > \sigma_{III}$$

In this case, three Mohr circles can be constructed (Figure 2.15): (circle (σ_I, σ_{II}) ; circle (σ_I, σ_{III}) ; circle $(\sigma_{II}, \sigma_{III})$)

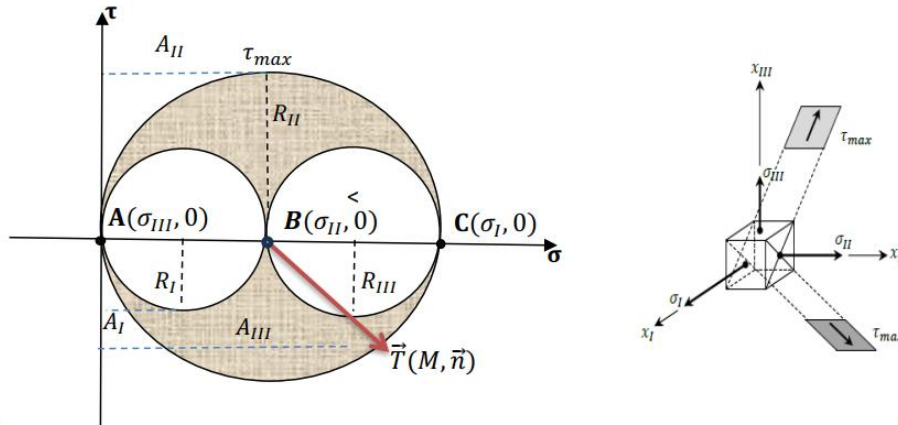


Figure 2.14 Description of Mohr's Circle from the principal stresses..

The equations of the three circles are:

$$\begin{aligned} \left(\sigma - \frac{\sigma_{II} + \sigma_{III}}{2} \right)^2 + \tau^2 &= \left(\frac{\sigma_{II} - \sigma_{III}}{2} \right)^2 \\ \left(\sigma - \frac{\sigma_I + \sigma_{III}}{2} \right)^2 + \tau^2 &= \left(\frac{\sigma_I - \sigma_{III}}{2} \right)^2 \\ \left(\sigma - \frac{\sigma_I + \sigma_{II}}{2} \right)^2 + \tau^2 &= \left(\frac{\sigma_I - \sigma_{II}}{2} \right)^2 \end{aligned}$$

Raduis of these circles are:

$$R_I = \frac{\sigma_{II} - \sigma_{III}}{2}; R_{II} = \frac{\sigma_I - \sigma_{III}}{2}; R_{III} = \frac{\sigma_I - \sigma_{II}}{2}$$

The largest circle corresponds to the pair (σ_I, σ_{III}) . Therefore, the maximum shear stress in three dimensions is: $\tau_{max} = \frac{\sigma_I - \sigma_{III}}{2}$.

The Mohr representation is an essential graphical tool in solid mechanics because it provides a clear visualization of stress transformations and principal stress states.

1. 5. Conclusion

In this chapter, the fundamental concepts of the theory of stress state have been introduced and developed. The stress tensor was presented as a mathematical tool used to describe the distribution of internal forces within a continuous medium. The Euler-Cauchy principle and Cauchy's theorem established the relationship between traction vectors and the stress tensor, while the notions of normal and shear stresses allowed a clearer understanding of the mechanical behavior of materials. The study of principal stresses, principal directions, and stress invariants provided important information about the critical stress conditions inside a solid body. Furthermore, the differential equations of equilibrium expressed the balance

between internal stresses, body forces, and inertial effects. Particular stress states such as uniaxial, plane, isotropic, and shear stress states were also examined. Finally, Mohr's circle offered a powerful graphical representation for stress transformation and the determination of principal and maximum shear stresses.

The concepts developed in this chapter form the basis for the next chapter devoted to the Theory of Strain State, where the deformation and displacement of materials under the action of stresses will be studied.

2. Exercises

Exercise 1

The stress state at a point in a solid body is defined by the following stress tensor:

$$\sigma = \begin{bmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix} (MPa)$$

1. Determine the principal stresses of the stress tensor σ .
2. Determine the traction vector acting at point M on the plane whose unit normal vector is:

$$\vec{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Then deduce the norm stress and the shear stress acting on the plane.

Exercise 2

In an elastic solid, the stress field is described by the following tensor:

$$\sigma = \begin{bmatrix} 4x_1^3x_3 & x_1 - x_3 & 4x_2^2 \\ x_1 - x_3 & -4x_2^2 & 0 \\ 4x_2^2 & 0 & 4x_1x_3^3 \end{bmatrix} (MPa)$$

1. Using the equilibrium equations, determine the expression of the body force vector:

$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

2. Write the stress tensor σ at the point $M(1,1,1)$

Exercise 3

Consider a solid body in equilibrium made of a linear elastic and isotropic material. The body is subjected over its entire boundary to surface forces of the form:

$$\vec{T}_d = -P\vec{n}$$

where: P is a constant pressure, $\vec{n}$ is the outward unit normal vector.

1. Determine the stress state inside the solid.

Exercise 4

The stress state at a point of a solid is defined by:

$$\sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4a & 6a \\ 0 & 6a & 4a \end{bmatrix} (MPa), \text{ where } a \text{ is a real constant.}$$

1. Determine the stress state for $a=0$
2. Determine, as functions of a , the principal stresses.
3. Draw the Mohr circle for $a=1$.

3. Solutions

Solution to exercise 1

The stress tensor is given by:

$$\sigma = \begin{bmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix} (MPa)$$

1. Determination of the Principal Stresses

The principal stresses are obtained from the characteristic equation:

$$\det(\sigma - \lambda I) = 0$$

Substituting the stress tensor:

$$\det \left(\begin{bmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

From the tensor, it is immediately observed that $\lambda = 8\text{Mpa}$ is one principal stress associated with the direction $\vec{e}_3$. The remaining two principal stresses are obtained from the reduced 2×2 matrix :

$$\det \begin{bmatrix} 6 - \lambda & 3 \\ 3 & 6 - \lambda \end{bmatrix} = 0$$

Expanding the determinant:

$$\begin{aligned} (6 - \lambda)^2 - 9 &= 0 \\ (6 - \lambda + 3)(6 - \lambda - 3) &= 0 \end{aligned}$$

Thus: $\lambda_1 = 9\text{Mpa}$; $\lambda_2 = 3\text{Mpa}$

Finally, the principal stresses are: $\sigma_I = 9\text{Mpa}$; $\sigma_{II} = 8\text{Mpa}$; $\sigma_{III} = 3\text{Mpa}$

2. Traction Vector Acting on the Plane

The unit normal vector is: $\vec{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$

The traction vector is calculated using Cauchy's relation:

$$\vec{T}(M, \vec{n}) = \sigma \cdot \vec{n}$$

$$\vec{T}(M, \vec{n}) = \begin{bmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{Bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{Bmatrix}$$

which gives:

$$\vec{T}(M, \vec{n}) = \begin{Bmatrix} \frac{9}{\sqrt{3}} \\ \frac{9}{\sqrt{3}} \\ \frac{8}{\sqrt{3}} \end{Bmatrix} \text{Mpa}$$

3. Normal and Shear Stresses

The normal stress is obtained by projecting the traction vector onto the normal direction:

$$\sigma_n = \vec{T} \left(\vec{M}, \vec{n} \right) \cdot \vec{n}$$

$$\sigma_n = \left\langle \frac{9}{\sqrt{3}}, \frac{9}{\sqrt{3}}, \frac{8}{\sqrt{3}} \right\rangle \cdot \begin{Bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{Bmatrix} = \frac{26}{3} = 8.66 \text{ Mpa}$$

The shear stress is determined from:

$$\tau = \sqrt{\|\vec{T}\|^2 - \sigma_n^2} = 0.45 \text{ Mpa}$$

Solution to Exercise 2

The stress tensor is:

$$\sigma = \begin{bmatrix} 4x_1^2 x_3 & x_1 - x_3 & 4x_2^2 \\ x_1 - x_3 & -4x_2^2 & 0 \\ 4x_2^2 & 0 & 4x_1 x_3^2 \end{bmatrix} \text{ (MPa)}$$

1. Determination of the Body Force Vector

Using the equilibrium equation: $\text{div}(\sigma) + f = 0$

we obtain:

$$\begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = 0 \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0 \end{cases}$$

After differentiation:

$$\begin{cases} 8x_1 x_3 + f_1 = 0 \\ 1 - 8x_2 + f_2 = 0 \\ 8x_1 x_3 + f_3 = 0 \end{cases}$$

Therefore, the body force vector becomes:

$$\vec{f} = \begin{Bmatrix} -8x_1 x_3 \\ 8x_2 - 1 \\ -8x_1 x_3 \end{Bmatrix}$$

2. Stress Tensor at Point M(1,1,1)

Substituting $x_1 = x_2 = x_3 = 1$

$$\sigma(M) = \begin{bmatrix} 4 & 0 & 4 \\ 0 & -4 & 0 \\ 4 & 0 & 4 \end{bmatrix} \text{ MPa}$$

Solution to Exercise 3

The surface force applied on the boundary is:

$$\vec{T}_d = -P \vec{n}$$

Using Cauchy's relation:

$$\sigma \cdot \vec{n} = \vec{T}_d$$

we obtain:

$$\vec{\sigma} \cdot \vec{n} = -P\vec{n}$$

This condition is satisfied only if the stress tensor is isotropic:

$$\sigma = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix}$$

Thus, the solid is subjected to a uniform hydrostatic compression state.

Solution to Exercise 4

The stress tensor is:

$$\sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4a & 6a \\ 0 & 6a & 4a \end{bmatrix} (MPa)$$

1. Stress State for $a = 0$

$$\sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This corresponds to a uniaxial stress state along the first direction.

2. Principal Stresses

The characteristic equation is:

$$\det(\sigma - \lambda I) = 0$$

which gives:

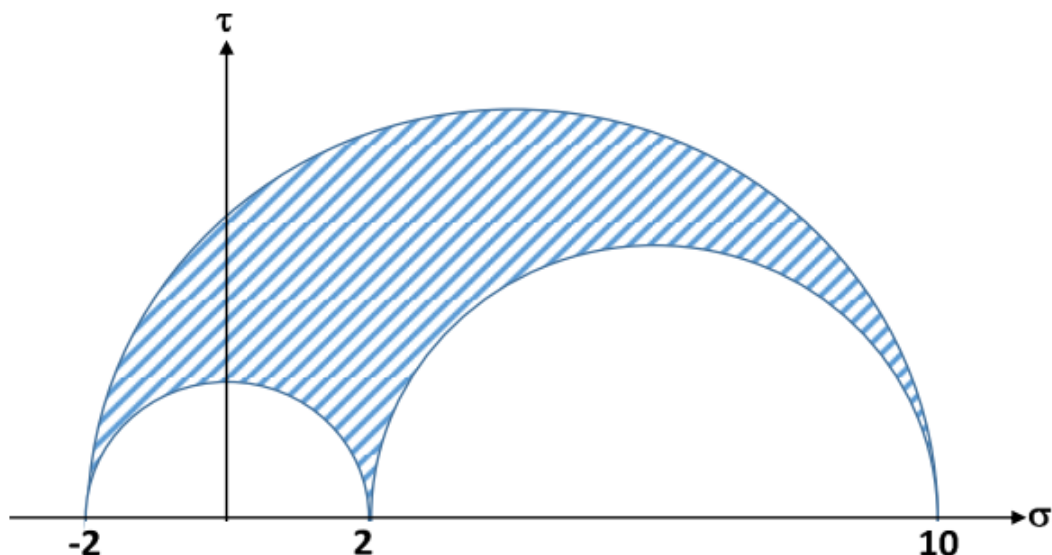
$$(2 - \lambda)[(4a - \lambda)^2 - (6a)^2] = 0$$

Solving: $\lambda_1 = 2$; $\lambda_2 = 10a$; $\lambda_3 = -2a$

Therefore, the principal stresses are: $\sigma_I = 10a$ MPa; $\sigma_{II} = 2$ Mpa; $\sigma_{III} = -2a$ MPa

3. Mohr Circle for $a=1$

For $a=1$, the principal stresses become: $\sigma_I = 10$ Mpa; $\sigma_{II} = 2$ Mpa; $\sigma_{III} = -2$ Mpa



Tricerle de Mohr

Chapter 3: Theory of Strain State

- 1. Introduction**
- 2. General concepts**
- 3. Strain tensor**
- 4. Relations between strains and displacements**
- 5. Principal strains and principal directions**
- 6. Geometrical representation**
- 7. Strain compatibility equations**
- 8. Measurement of strains**
- 9. Conclusion**

1. Introduction

The theory of strain state is an essential part of elasticity and solid mechanics. When a body is subjected to external forces, its shape and dimensions change. These changes are called deformations or strains. The study of strain helps engineers understand how structures behave under loading conditions and whether they remain safe during service.

In elasticity theory, strain analysis is closely related to displacement and stress analysis. While displacement describes the movement of points inside a body, strain measures the relative deformation between neighboring points. The theory of strain state, therefore, provides the mathematical and physical tools necessary to describe the deformation of solid bodies.

This chapter introduces the main concepts of strain, the strain tensor, the relations between strains and displacements, principal strains, compatibility equations, and experimental methods used to measure strains. The presentation is simplified and adapted for engineering students studying elasticity and continuum mechanics.

2. General concepts

2.1 Kinematic Description (Lagrangian and Eulerian Approaches)

2.1.1 Configuration of a Continuous Medium

Consider a continuous body occupying a region D of the Euclidean space E at a given instant t . A continuous medium S is defined as a set of material particles whose positions evolve over time.

The set of all positions occupied by the particles of S at time t is called the **configuration** of the body and is denoted by C . The transition from one configuration, C_0 at the initial time t_0 to another configuration C at time t represents the deformation or motion of the medium. The initial configuration C_0 is generally chosen as the **reference configuration**.

Let the Cartesian reference be defined by: $R(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$. For a material particle M belonging to the body C :

- In the reference configuration C_0 at time t_0 : $\vec{X} = \vec{OM}_0$
- In the current configuration C at time t : $\vec{x} = \vec{OM}$

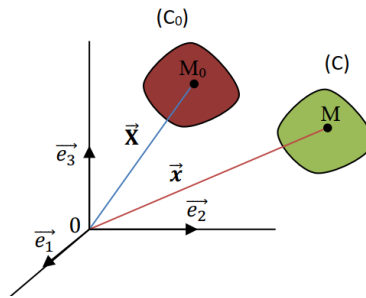


Figure 3.1

where: $\vec{X}$ denotes the initial position vector, $\vec{x}$ denotes the current position vector after deformation.

2.1.2 Lagrangian Description

In the Lagrangian description, the motion of the body is studied by tracking each material particle from its initial position. Thus, the current position of any particle at time t is expressed as a function of its original position at time t_0 .

The motion is described by the transformation:

$$\vec{x} = \vec{\varphi}(\vec{X}, t) = \vec{x} = \varphi(X_1, X_2, X_3, t)$$

where: φ is a bijective vector-valued mapping, X_1, X_2, X_3 are the material coordinates of the particle in the reference configuration, t represents time.

This approach is widely employed in solid mechanics because it naturally follows the deformation history of each material point.

2.1.3 Eulerian Description

In the Eulerian description, physical quantities are defined with respect to the current spatial position of the particles rather than their original locations. The observer focuses on fixed points in space and studies how the material flows through them.

For example, the velocity field is expressed as:

$$\vec{V} = \vec{V}(\vec{x}, t)$$

or in component form:

$$\vec{V} = V(x_1, x_2, x_3, t)$$

where x_1, x_2, x_3 : are the spatial coordinates, t is time.

These variables are known as **Eulerian variables**, **spatial variables**, or **geometrical variables**. This formulation is particularly suitable in fluid mechanics, where the interest lies in describing the flow behavior at fixed locations in space.

2.2 Study of Deformation Using Lagrangian Variables

2.2.1 Deformation Gradient Tensor

Consider an infinitesimal material (Figure 3.2) vector represented by:

- In the reference configuration C_0 :

$$\vec{X} = \vec{M}_0 \vec{M}'_0$$

- In the current configuration C :

$$\vec{dx} = \vec{M} \vec{M}'$$

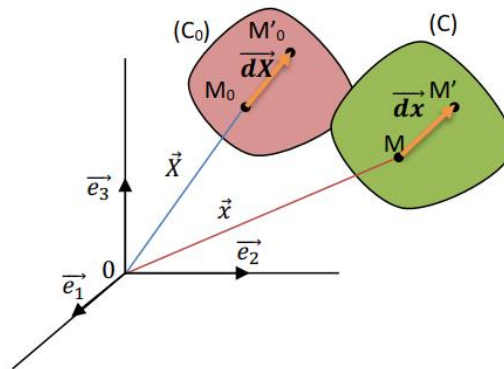


Figure 3.2

The displacement vector of a particle is defined as:

$$\vec{U}(\vec{X}, t) = \vec{x} - \vec{X}$$

Suppose that the neighboring particle M'_0 in the reference configuration moves to M' in the current configuration. $\vec{OM}'_0 = \vec{OM}_0 + \vec{M}_0 \vec{M}'_0$

Then:

$$\vec{X}' = \vec{X} + \vec{dX}$$

And

$$\vec{x}' = \vec{\varphi}(\vec{X} + \vec{dX}, t)$$

Using a first-order Taylor expansion:

$$x'_i = \varphi_i(\vec{X}, t) + \frac{\partial \varphi_i}{\partial X_j} dX_j$$

In vector notation:

$$\vec{x}' = \vec{x} + \nabla \varphi(\vec{X}, t) d\vec{X}$$

Hence:

$$d\vec{x} = \nabla \varphi(\vec{X}, t) d\vec{X}$$

The tensor:

$$F(\vec{X}, t) = \nabla \varphi(\vec{X}, t)$$

called the deformation gradient tensor.

Its components are given by:

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

Therefore, $d\vec{x} = F d\vec{X}$

The deformation gradient tensor characterizes the local deformation of the body around a material point and contains information about stretching, rotation, and distortion.

2.2.2 Green-Lagrange Strain Tensor

Deformation measures the variation of lengths and angles between material elements during motion. It can therefore be characterized through the change in the scalar product of two infinitesimal vectors between the configurations C_0 and C .

Let:

$$d\vec{x} = F d\vec{X}$$

And

$$\delta \vec{x} = F \delta \vec{X}$$

The scalar product in the deformed configuration becomes:

$$d\vec{x} \cdot \delta \vec{x} = d\vec{X} \cdot (F^T F) \delta \vec{X}$$

Define the tensor:

$$C = F^T F$$

where C is known as the right Cauchy-Green deformation tensor or the dilatation tensor.

Thus:

$$C = F^T F$$

The variation of the scalar product between the initial and deformed configurations is:

$$d\vec{x} \cdot \delta \vec{x} - d\vec{X} \cdot \delta \vec{X} = 2 d\vec{X} \cdot E \delta \vec{X}$$

where E is the **Green-Lagrange strain tensor**, defined by:

$$E = \frac{1}{2}(C - I)$$

or equivalently,

$$E = \frac{1}{2}(F^T F - I)$$

In index notation:

$$E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij})$$

The Green–Lagrange strain tensor is a fundamental measure of finite deformation and is particularly suitable for large displacement and large strain analyses in continuum mechanics.

3. Strain tensor

3.1 Small Perturbation Hypothesis (SPH)

In the theory of linear elasticity, it is generally assumed that the body undergoes only very small deformations compared with its initial dimensions. This assumption is called the **Small Perturbation Hypothesis (SPH)** or the **Hypothesis of Small Displacements and Small Strains**.

Under this hypothesis, the current configuration CCC remains very close to the initial configuration C_0 : $C \approx C_0$

The displacement field and its spatial gradients remain very small compared with unity:

$$\|\vec{U}\| \ll 1 \text{ and } \|\vec{\nabla U}\| \ll 1$$

or equivalently:

$$|U_i| \ll 1 \text{ and } |U_{i,j}| \ll 1$$

Means $\|\vec{U}\| \ll 1$ and $\|\vec{\nabla U}\| \ll 1$

Consequently:

- The coordinates in the deformed and undeformed configurations are nearly identical:

$$x_i \approx X_i$$

Nonlinear terms involving products of displacement gradients are neglected.

Therefore, the Green-Lagrange strain tensor simplifies into the **linearized strain tensor**, also called the **infinitesimal strain tensor**.

3.2 Linearized Strain Tensor

For small deformations, the strain tensor becomes:

$$\varepsilon = \frac{1}{2}(\vec{\nabla U} + (\vec{\nabla U})^T)$$

In matrix form:

$$\varepsilon = \begin{bmatrix} \frac{\partial U_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}\right) & \frac{\partial U_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial U_2}{\partial x_3} + \frac{\partial U_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial U_2}{\partial x_3} + \frac{\partial U_3}{\partial x_2}\right) & \frac{\partial U_3}{\partial x_3} \end{bmatrix}$$

In index writing is : $\varepsilon_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$

where:

- U_1, U_2, U_3 are the displacement components,
- x_1, x_2, x_3 are the spatial coordinates,
- ε_{ij} are the components of the infinitesimal strain tensor.

Because the tensor is symmetric:

$$\varepsilon_{ij} = \varepsilon_{ji}$$

only six independent strain components exist.

3.3 Normal and Shear Strains

The diagonal terms of the strain tensor represent the **normal strains**:

$$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$$

These components describe elongation or contraction along the principal coordinate axes.

The off-diagonal terms represent the **shear strains**:

$$\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$$

Engineering shear strains are defined as:

$$\gamma_{12} = 2\varepsilon_{12}, \quad \gamma_{13} = 2\varepsilon_{13}, \quad \gamma_{23} = 2\varepsilon_{23}$$

In index writing is: $\gamma_{ij} = 2\varepsilon_{ij}$

3.4 Linearized Extension and Shear Relations

For a material direction represented by the unit vector $\vec{n}_0$, the normal strain becomes:

$$\varepsilon(\vec{n}_0) = n_{0i} \varepsilon_{ij} n_{0j}$$

Special cases:

$$\varepsilon(\vec{e}_1) = \varepsilon_{11}, \quad \varepsilon(\vec{e}_1) = \varepsilon_{11}, \quad \varepsilon(\vec{e}_2) = \varepsilon_{22}$$

Similarly, the angular distortion between two initially orthogonal directions $\vec{n}_0$ and $\vec{m}_0$ is:

$$\gamma(\vec{n}_0, \vec{m}_0) = 2n_{0i} \varepsilon_{ij} m_{0j}$$

For the Cartesian basis vectors:

$$\gamma(\vec{e}_1, \vec{e}_2) = 2\varepsilon_{12}, \quad \gamma(\vec{e}_1, \vec{e}_2) = 2\varepsilon_{12}, \quad \gamma(\vec{e}_2, \vec{e}_3) = 2\varepsilon_{23}$$

These relations are fundamental in the analysis of deformation in solid mechanics.

4. Relations between strains and displacements

The deformation of a solid body is completely described by the displacement field of its material points. In elasticity theory, strains are obtained from the spatial derivatives of the displacement components. Therefore, the relationship between strain and displacement constitutes the fundamental kinematic basis of continuum mechanics.

For infinitesimal deformations, the strain components are linearly related to the displacement gradients. These relations depend on the coordinate system used to describe the geometry of the problem. The common coordinate systems in engineering applications are (Figure 3.3):

- Cartesian coordinates (x,y,z),
- Cylindrical coordinates (r,θ,z),
- Spherical coordinates (r, θ, φ).

Each coordinate system is particularly suitable for specific structural geometries:

- Cartesian coordinates → beams, plates, rectangular solids,
- Cylindrical coordinates → pipes, shafts, pressure vessels,
- Spherical coordinates → spherical tanks, domes, planets.

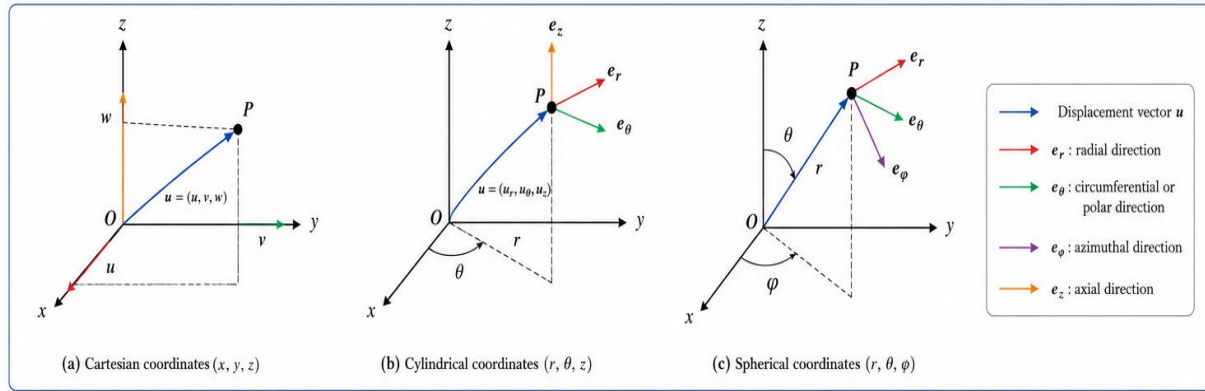


Figure 3.3 Common coordinate systems.

4.1 Strain–Displacement Relations in Cartesian Coordinates

Cartesian coordinates are the simplest and most widely used coordinate system in elasticity. The displacement vector is written as:

$$\vec{u} = (u, v, w)$$

where: u is displacement along the x -direction, v is displacement along the y -direction, w is displacement along the z -direction.

a) Normal Strains

The normal strains represent elongation or contraction along the coordinate axes.

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

These equations indicate that normal strain is simply the spatial rate of change of displacement in the same direction.

Physical Interpretation : Positive strain $\rightarrow$ extension, Negative strain $\rightarrow$ compression,

b) Shear Strains

The shear strains measure the variation of angles between initially perpendicular material lines.

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Physical Meaning : Shear strains produce angular distortion without necessarily changing the volume of the material.

c) Tensor Form

The infinitesimal strain tensor in Cartesian coordinates is:

$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{xy}}{2} & \epsilon_{yy} & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & \epsilon_{zz} \end{bmatrix}$$

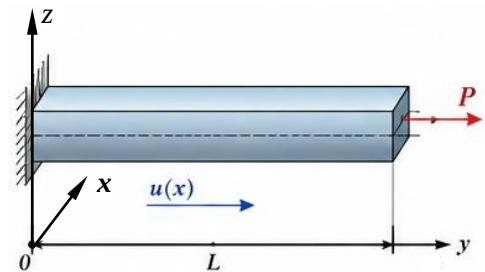


Figure 3.4 Axially loaded bar.

4.2 Strain–Displacement Relations in Cylindrical Coordinates

Spherical coordinates are used when the geometry possesses spherical symmetry.

The coordinates are: (r, θ, ϕ)

where: r is radial distance, θ is polar angle, ϕ in azimuthal angle.

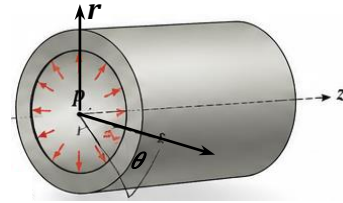
The displacement field becomes:

$$\vec{u} = (u_r, u_\theta, u_\phi)$$

a) Normal Strains

- Radial Strain : $\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$
- Polar Strain : $\varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$
- Azimuthal Strain : $\varepsilon_{\phi\phi} = \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r}$

Figure 3.5 Thick cylindrical pressure vessel.



The additional geometric terms arise from the curvature of spherical coordinates.

b) Shear Strains

- Radial-Azimuthal Shear : $\gamma_{r\phi} = \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}$
- Radial-Azimuthal Shear : $\gamma_{r\phi} = \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}$
- Polar-Azimuthal Shear : $\gamma_{\theta\phi} = \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi \cot \theta}{r}$

c) Engineering Applications

Cylindrical strain relations are extensively used in:

- torsion of shafts,
- pressure vessel analysis,
- pipeline mechanics,
- rotating machinery,
- tunnel engineering.

4.3 Strain-Displacement Relations in Spherical Coordinates

Spherical coordinates are used when the geometry possesses spherical symmetry.

The coordinates are: (r, θ, ϕ)

where: r is radial distance, θ is polar angle, ϕ is azimuthal angle.

The displacement field becomes:

$$\vec{u} = (u_r, u_\theta, u_\phi)$$

a) Normal Strains

- Radial Strain : $\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$
- Polar Strain : $\varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$
- Azimuthal Strain: $\varepsilon_{\phi\phi} = \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r}$

b) Shear Strains

- Radial-Polar Shear : $\gamma_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$
- Radial-Azimuthal Shear : $\gamma_{r\phi} = \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}$
- Polar-Azimuthal Shear : $\gamma_{\theta\phi} = \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi \cot \theta}{r}$

c) Engineering Applications

Spherical coordinate strain relations are used in:

- spherical pressure vessels,
- geomechanics,
- astrophysics,

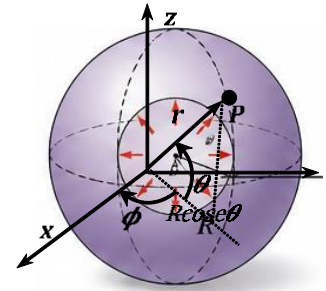


Figure 3.6 Spherical shell.

- biomechanics,
- shell structures,
- seismic wave propagation.

5 Principal strains and principal directions

5.1 Principal Strains

Consider an orthonormal basis: $B(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$. The strain tensor is written as:

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

Since the tensor is symmetric: $\varepsilon_{ij} = \varepsilon_{ji}$

there exists a principal coordinate system in which the tensor becomes diagonal:

$$\varepsilon = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

The quantities:

$$\varepsilon_1, \varepsilon_2, \varepsilon_3$$

called the **principal strains**.

They are obtained by solving the characteristic equation:

$$\det(\varepsilon - \lambda I) = 0$$

which leads to the cubic polynomial:

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$$

where:

I_1, I_2 and I_3 are the invariants of the strain tensor.

Generally:

$$\varepsilon_1 > \varepsilon_2 > \varepsilon_3$$

The principal strains correspond to the maximum and minimum normal deformations at the point.

5.2 Principal Directions

The principal directions are obtained by solving:

$$(\varepsilon - \lambda I)\vec{X} = 0$$

where:

$\vec{X}$ is a principal direction vector,
 λ is a principal strain.

The principal directions are mutually orthogonal because the strain tensor is symmetric.

5.3 Decomposition into Spherical and Deviatoric Parts

The strain tensor can be decomposed into:

- a spherical (volumetric) part,
- a deviatoric (distortional) part.

Thus:

$$\varepsilon = \varepsilon_m I + e$$

where spherical part is

$$\varepsilon_m = \frac{1}{3} \text{tr}(\varepsilon)$$

And deviatoric part is

$$e = \varepsilon - \varepsilon_m I$$

Physical Interpretation

- The spherical part produces volume change.
- The deviatoric part produces distortion without changing volume.

4. This decomposition is very important in elasticity, plasticity, and failure theories.

5. 6. Geometrical representation

Introduce Mohr's circle as a graphical method used to represent the strain state at a point.

Explain that it simplifies the transformation of strains and allows the determination of:

- principal strains,
- maximum shear strain,
- strain components on inclined planes.

6.1 Mohr's Circle for Plane Strain

In the case of plane deformation, the strain state at a point is completely defined by three components:

$$\varepsilon_x, \varepsilon_y, \gamma_{xy}$$

The associated strain tensor is written as:

$$[\varepsilon] = \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} \\ \frac{\gamma_{xy}}{2} & \varepsilon_y \end{bmatrix}$$

Mohr's circle is constructed in a coordinate system where the horizontal axis represents the normal strain (ε), while the vertical axis represents half of the shear strain ($\gamma/2$) (Figure 3.7).

The center of the circle is located at:

$$C = \frac{\varepsilon_x + \varepsilon_y}{2}$$

The radius of the circle is:

$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

The geometrical construction of the circle allows the determination of the principal strains, which correspond to the intersections of the circle with the horizontal axis:

$$\varepsilon_1 = C + R, \quad \varepsilon_2 = C - R$$

At these points, the shear strain becomes zero, meaning that the corresponding planes are principal planes.

The orientation of the principal directions is obtained from:

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

The maximum shear strain corresponds to the upper and lower points of the circle and is expressed by:

$$\gamma_{\max} = 2R \quad \text{or} \quad \gamma_{\max} = \varepsilon_1 - \varepsilon_2$$

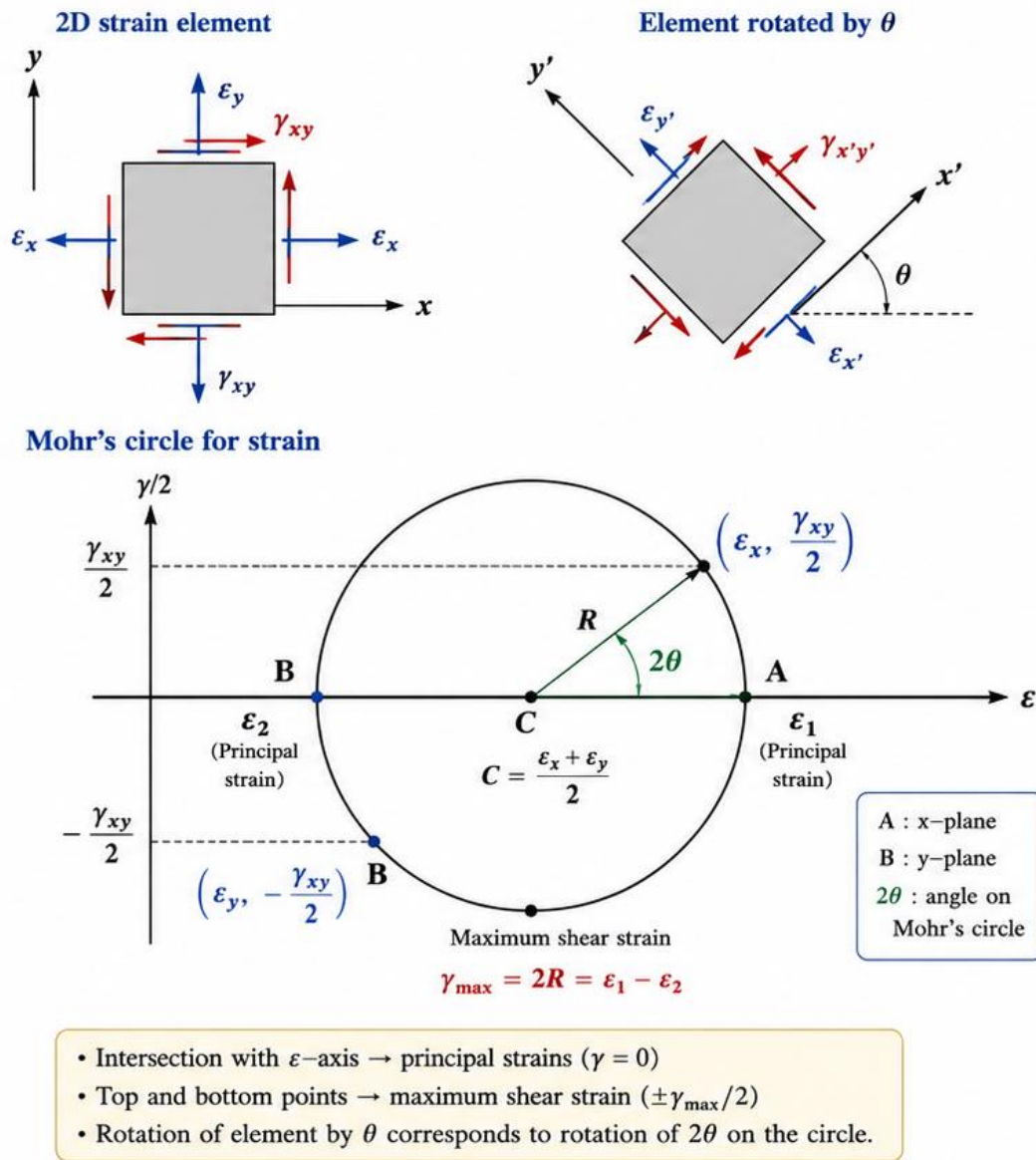


Figure 3.7 Mohr's Circle for Plane Strain.

An important property of Mohr's circle is that a physical rotation of the material element by an angle θ corresponds to a rotation of 2θ on the circle. This graphical interpretation considerably simplifies strain transformation analysis.

Mohr's circle is extensively used in structural mechanics, material science, fracture mechanics, and finite element analysis because it provides a direct visualization of the deformation state inside materials.

6.2 Mohr's Circles for Three-Dimensional Strain State

In three-dimensional problems, the strain tensor contains three principal strains:

$$\epsilon_1, \epsilon_2, \epsilon_3$$

generally ordered as: $\epsilon_1 > \epsilon_2 > \epsilon_3$

The geometrical representation of the strain state in three dimensions requires the construction of three Mohr's circles instead of a single one (Figure 3.8). These circles are formed using the following pairs of principal strains: (ϵ_1, ϵ_2) , (ϵ_2, ϵ_3) , (ϵ_1, ϵ_3) . Each circle represents the strain transformation between two principal directions.

The centers of the circles are: $C_{12} = \frac{\varepsilon_1 + \varepsilon_2}{2}$, $C_{23} = \frac{\varepsilon_2 + \varepsilon_3}{2}$, $C_{13} = \frac{\varepsilon_1 + \varepsilon_3}{2}$

The radii are given by:

$$R_{12} = \frac{\varepsilon_1 - \varepsilon_2}{2}, R_{23} = \frac{\varepsilon_2 - \varepsilon_3}{2}, R_{13} = \frac{\varepsilon_1 - \varepsilon_3}{2}$$

The largest circle corresponding to $(\varepsilon_1, \varepsilon_3)$, represents the maximum shear strain in the material;

$$\gamma_{\max} = \varepsilon_1 - \varepsilon_3$$

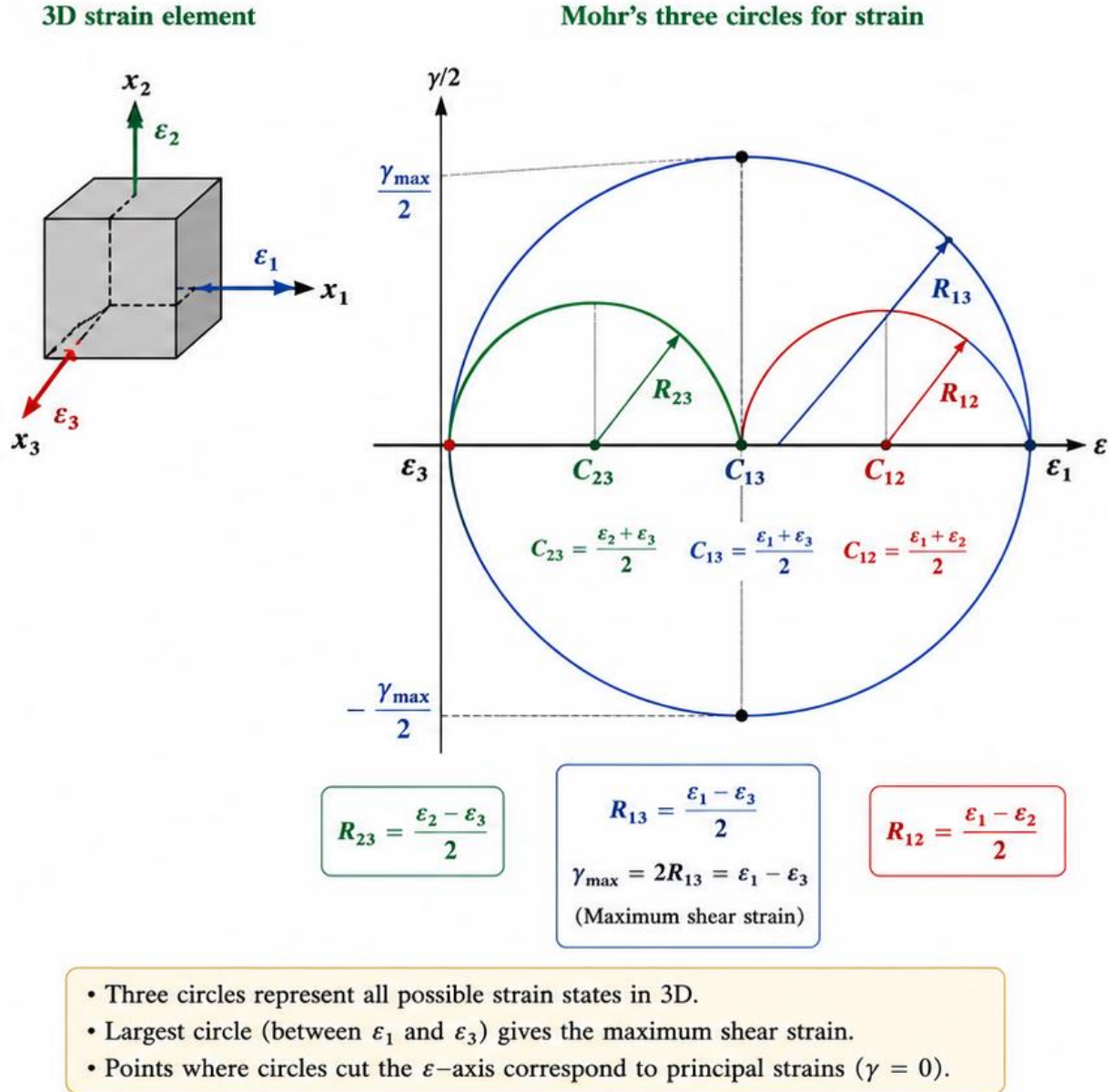


Figure 3.8 Mohr's Circles for Three-Dimensional Strain State.

In three-dimensional analysis, Mohr's circles provide a complete graphical description of all possible strain states acting at a point. They are particularly useful in advanced elasticity, plasticity, geomechanics, and material failure theories. The graphical representation also helps engineers identify the most critical deformation directions and understand complex multiaxial loading conditions.

Applications of three-dimensional Mohr's circles include:

- pressure vessel analysis,

- aerospace structures,
- underground structures,
- geotechnical engineering,
- finite element simulations,
- failure prediction of materials.

7. Strain compatibility equations

The infinitesimal strain tensor is defined by:

$$\varepsilon_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$$

If the displacement field $U(X, t)$ is known, the strain tensor can always be calculated.

However, the inverse problem is more complex: if the strain tensor is known, can we always determine a displacement field?

The answer is not always. The strain field must satisfy certain conditions called the compatibility equations.

A strain field is said to be compatible if there exists a continuous and single-valued displacement field associated with it. For infinitesimal deformations, the compatibility equations are:

$$\begin{aligned}\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} &= 0 \\ \varepsilon_{11,23} + \varepsilon_{23,11} - \varepsilon_{31,21} - \varepsilon_{12,31} &= 0 \\ \varepsilon_{22,33} + \varepsilon_{33,22} - 2\varepsilon_{23,23} &= 0 \\ \varepsilon_{22,31} + \varepsilon_{31,22} - \varepsilon_{12,32} - \varepsilon_{23,12} &= 0 \\ \varepsilon_{22,31} + \varepsilon_{31,22} - \varepsilon_{12,32} - \varepsilon_{23,12} &= 0 \\ \varepsilon_{33,11} + \varepsilon_{11,33} - 2\varepsilon_{31,31} &= 0 \\ \varepsilon_{33,12} + \varepsilon_{12,33} - \varepsilon_{23,13} - \varepsilon_{31,23} &= 0\end{aligned}$$

These equations guarantee the continuity of the material after deformation.

8. Measurement of strains

Theoretical formulations in elasticity and continuum mechanics make it possible to determine the distribution of strains inside deformable bodies. However, in practical engineering applications, it is often necessary to measure strains experimentally in order to validate analytical models and numerical simulations. Experimental strain measurements also provide essential information about the mechanical behavior of structures subjected to external loads.

Strain measurement techniques are widely used in structural engineering, aerospace engineering, mechanical design, and material science. They allow engineers to monitor the integrity of structures, detect the initiation of cracks, evaluate fatigue damage, and estimate stresses indirectly through constitutive relations.

In civil engineering, strain measurements are commonly performed on bridges, dams, pipelines, and high-rise buildings in order to ensure structural safety during service conditions. In aerospace applications, strain sensors are installed on aircraft wings and fuselage components to study deformation under aerodynamic loading. Similarly, in mechanical

engineering, strain measurements are employed to analyze the behavior of machine components subjected to complex loading conditions.

Modern strain measurement methods range from simple mechanical devices to highly advanced optical and electronic systems capable of providing extremely accurate full-field deformation measurements.

8.2 Mechanical Extensometers

Mechanical extensometers are among the oldest and simplest devices used for strain measurement. These instruments determine strain by measuring the variation in length between two points of a deformable specimen during loading (figure 3.9).



Figure 3.9 Mechanical Extensometer.

The normal strain is defined as: $\varepsilon = \frac{\Delta L}{L}$

where L is the initial length and ΔL represents the change in length produced by deformation. In tensile testing, extensometers are generally attached directly to the specimen in order to measure elongation with high precision. Clip-on extensometers are widely used in laboratories because they are easy to install and provide accurate displacement measurements over small gauge lengths.

Mechanical extensometers possess several advantages, including simplicity of operation, low cost, and satisfactory accuracy for standard laboratory tests. Nevertheless, these devices are limited to local measurements and are generally unsuitable for high-speed dynamic loading or large deformations.

Despite the development of advanced electronic and optical methods, extensometers remain widely employed in mechanical testing laboratories because of their reliability and ease of use.

8.3 Electrical Resistance Strain Gauges

Electrical resistance strain gauges are the most commonly used sensors for experimental strain measurements. Their operating principle is based on the variation of electrical resistance caused by mechanical deformation.

When the material deforms, the strain gauge bonded to its surface undergoes the same deformation, producing a change in its electrical resistance. This variation is proportional to the strain experienced by the structure.

The sensitivity of a strain gauge is characterized by the gauge factor: $GF = \frac{\Delta R/R}{\varepsilon}$

where ΔR is the resistance variation, and R is the initial electrical resistance.

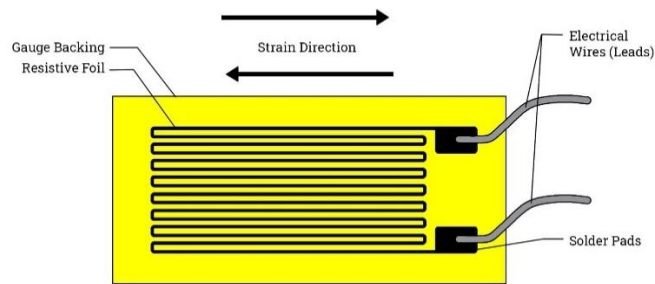


Figure 3.10 Electrical Resistance Strain Gauge.

In practical applications, strain gauges are generally connected to a Wheatstone bridge circuit, which converts very small resistance changes into measurable voltage variations. The resulting electrical signal is then amplified and processed using data acquisition systems.

Electrical resistance strain gauges offer numerous advantages, such as high sensitivity, compact dimensions, relatively low cost, and easy installation. Consequently, they are extensively used in structural analysis, fatigue testing, experimental mechanics, and vibration analysis.

However, accurate measurements require careful installation procedures because improper bonding, temperature variations, and electrical noise may introduce significant experimental errors.

8.4 Strain Gauge Rosettes

In many engineering situations, the principal strain directions are not known in advance. Under such conditions, a single strain gauge cannot provide sufficient information to completely characterize the strain state. For this reason, several strain gauges are combined into configurations known as strain gauge rosettes.

A strain rosette consists of two or three strain gauges mounted at different orientations on the same surface. The most common configurations are the rectangular rosette, the delta rosette, and the three-element rosette.

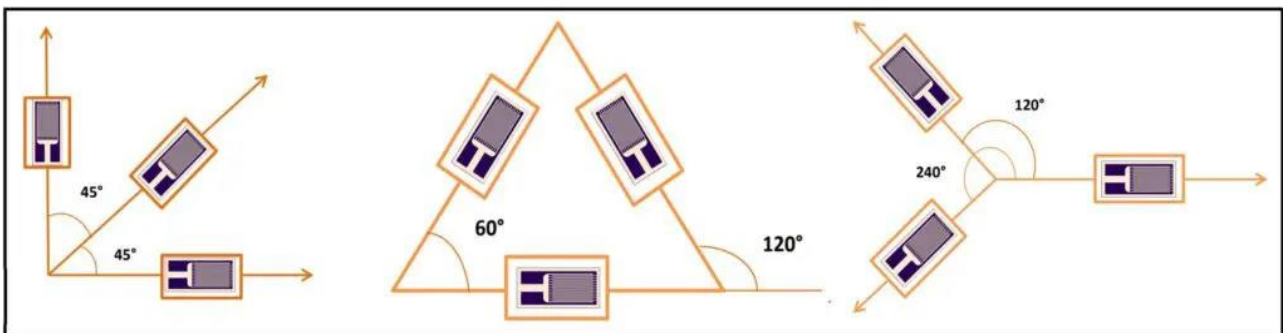


Figure 3.10 Strain Gauge Rosette configuration.

By combining the measurements obtained from different orientations, it becomes possible to determine the principal strains, principal directions, and maximum shear strain at a point. Strain gauge rosettes are extensively used in experimental stress analysis, especially in regions subjected to complex multiaxial loading conditions.

8.5 Optical Methods for Strain Measurement

Recent advances in experimental mechanics have led to the development of non-contact optical methods capable of measuring strain fields with very high precision. Unlike conventional strain gauges, optical techniques provide full-field measurements over large surfaces.

Among the most widely used optical methods are Digital Image Correlation (DIC), laser interferometry, moiré techniques, and photoelasticity.

Digital Image Correlation is based on tracking the displacement of a random speckle pattern applied to the surface of the specimen (Figure 3.11). By comparing images captured before and after deformation, displacement and strain fields can be computed with high accuracy.

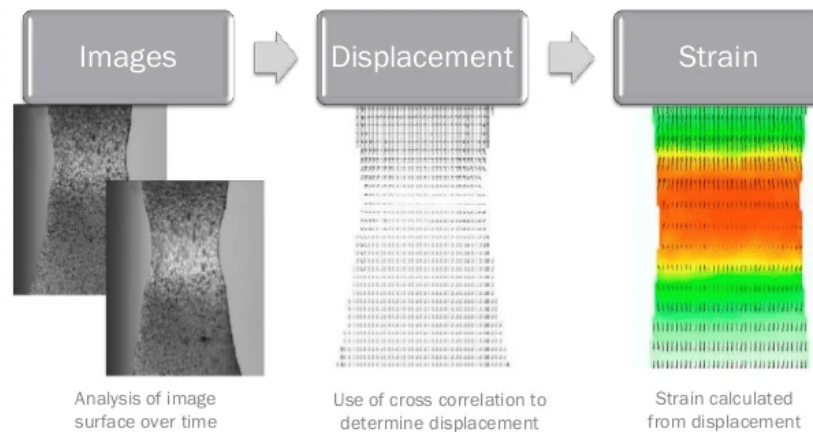


Figure 3.11 Digital Image Correlation.

Optical techniques possess several important advantages. They eliminate physical contact with the specimen, provide full-field strain distributions, and are suitable for complex geometries and large deformations. Consequently, these methods are extensively used in research laboratories, aerospace engineering, and advanced material characterization.

8.6 Fiber Optic Sensors

Fiber optic sensors constitute one of the most advanced technologies for strain measurement and structural health monitoring. These sensors use light propagation inside optical fibers to detect mechanical deformation.

Among the different technologies, Fiber Bragg Grating (FBG) sensors are the most widely used. In these sensors, deformation modifies the reflected wavelength of light, allowing strain to be determined with high precision.

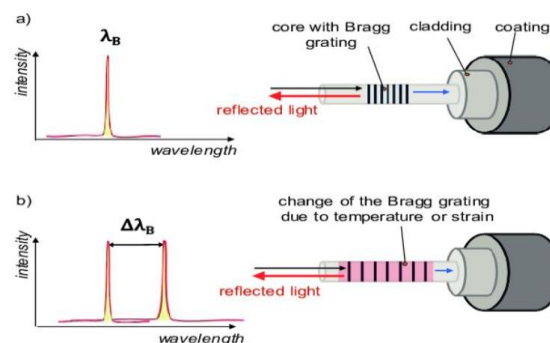


Figure 3.12 Fiber Bragg Grating sensor.

Fiber optic sensors present numerous advantages, including immunity to electromagnetic interference, high sensitivity, long-distance monitoring capability, and resistance to harsh environmental conditions.

These characteristics make fiber optic sensors particularly suitable for monitoring bridges, pipelines, dams, aircraft structures, and smart composite materials.

9. Conclusion

This chapter presented the fundamental principles of the theory of strain state used to describe the deformation of continuous media under mechanical loading. The kinematic descriptions of motion using the Lagrangian and Eulerian approaches were introduced, followed by the deformation gradient tensor and the Green–Lagrange strain tensor. Under the small perturbation hypothesis, the linearized strain tensor and the definitions of normal and shear strains were established. The strain–displacement relations were developed in Cartesian, cylindrical, and spherical coordinate systems for different engineering geometries. Principal strains, principal directions, and the decomposition into spherical and deviatoric parts were also discussed to better understand volumetric and distortional deformations. Mohr’s circles provided a graphical interpretation of strain transformations in both two-dimensional and three-dimensional cases, while compatibility equations ensured the continuity of deformation fields. Finally, several experimental techniques for strain measurement, including extensometers, strain gauges, optical methods, and fiber optic sensors, were presented, highlighting the strong connection between theoretical analysis and practical engineering applications.

Exercises

Exercise 1 — Displacement Field and Cauchy Deformation Tensor

A homogeneous continuous medium undergoes the following transformation:

$$\begin{cases} x = x_0 + 2ay_0 \\ y = y_0 \\ z = z_0 \end{cases}$$

Determine:

2. The displacement vector $\{u\}$
3. The deformation gradient tensor $[F]$
4. The Cauchy deformation tensor $[C]$

Exercise 2 — Green-Lagrange Strain Tensor

Consider a motion described by the following equations:

$$\begin{cases} x = x_0 + \frac{y_0}{3} \\ y = y_0 \\ z = z_0 \end{cases}$$

Calculate:

- The deformation gradient tensor $[F]$
- The Cauchy deformation tensor $[C]$
- The stretches λ in the three directions (x,y,z)
- The angle between the axes (x,y)
- The Green–Lagrange strain tensor $[E]$

- The stretch and Green–Lagrange strain in the direction

$$\{n\} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

- The infinitesimal strain tensor $[\varepsilon]$

Exercise 3— Rotations and Principal Strains

A homogeneous and isotropic elastic medium undergoes a planar configuration change described in two coordinate systems as follows:

Reference Frame (1) : $\begin{cases} x = ax_0 \\ y = (2 - a)y_0 \end{cases}$

Reference Frame (2) : $\begin{cases} x = x_0 + 2by_0 \\ y = y_0 \end{cases}$ The parameters a and b are positive constants.

- 1) Determine for both cases:
 - a) The displacement field (u)
 - b) The rotation tensor (Ω)
 - c) The linearized strain tensor (ε)
 - d) The scalar invariants of strain
 - e) The principal strains
- 2) Deduce the relation between the constants a and b

Solutions

Solution to exercise 1

- 1) Displacement Vector

$$\begin{aligned} \{u\} &= \{x\} - \{x_0\} \\ u &= \begin{pmatrix} x_0 + 2ay_0 \\ y_0 \\ z_0 \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 2ay_0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

- 2) Deformation Gradient Tensor

$$[F] = \begin{bmatrix} 1 & 2a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- 3) Cauchy Deformation Tensor

$$\begin{aligned} [C] &= [F]^T [F] = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ [C] &= \begin{bmatrix} 1 & 2a & 0 \\ 2a & 4a^2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Solution to exercise 2

Given the motion:

$$\begin{cases} x = x_0 + \frac{y_0}{3} \\ y = y_0 \\ z = z_0 \end{cases}$$

- 1) Deformation Gradient Tensor

$$[F] = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This tensor completely describes the local transformation of the material.

2) Cauchy Deformation Tensor

$$[C] = [F]^T [F] = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 1 & 2a & 0 \\ 2a & 4a^2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3) Stretches in the Three Directions

$$\lambda(x) = C_{xx} = 1, \quad \lambda(y) = C_{yy} = \frac{10}{9}, \quad \lambda(z) = C_{zz} = 1$$

4) Angle Between the Axes (x, y)

The angle variation between two initially perpendicular axes is determined using:

$$\gamma(\vec{x}, \vec{y}) = \arccos\left(\frac{C_{xy}}{\sqrt{C_{xx}C_{yy}}}\right)$$

Substitute the components:

$$\gamma = \arccos\left(\frac{\frac{1}{3}}{\sqrt{1 \times \frac{10}{9}}}\right) \approx 0.32 \text{ rad}$$

$$\gamma \approx 18.43^\circ$$

This indicates that the initially orthogonal axes become distorted after deformation.

5) Green-Lagrange Strain Tensor

$$[E] = \frac{1}{2}([C] - [I]) = \frac{1}{2} \left(\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{10}{9} & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{18} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This tensor measures finite deformation while accounting for nonlinear geometric effects.

6) Stretch and Green-Lagrange Strain in Direction n

$$n = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

a) Stretch in Direction n

$$\lambda(n) = n^T C = \frac{1}{3} (1 \ 2 \ 2) \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{10}{9} & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\lambda(n) = 1.09$$

B) Green-Lagrange Strain in Direction n

$$\varepsilon_{GL}(n) = n^T E n = \varepsilon_{GL}(n) = \frac{1}{3} (1 \ 2 \ 2) \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{18} & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\varepsilon_{GL}(n) = 0.098$$

Solution to exercise 3

Chapter 4: Constitutive Laws

- 1. Introduction**
- 2. Generalized Hooke's Law**
- 3. Effect of temperature**
- 4. Strain energy**
- 5. Conclusion**

1. Introduction

Constitutive laws represent the fundamental relationships between stresses and strains in deformable materials and constitute one of the main pillars of continuum mechanics and elasticity theory. These laws describe the mechanical response of materials subjected to external loads, thermal variations, or other physical effects. In civil engineering, constitutive models are essential for predicting the behavior of structures such as beams, columns, bridges, pavements, and concrete elements under service conditions. Among the most important constitutive relations, Hooke's law provides the basis for linear elastic analysis of isotropic materials. The incorporation of thermal effects allows engineers to evaluate thermal deformations and stresses generated by temperature changes. Furthermore, the concept of strain energy plays a crucial role in structural stability, energy methods, and finite element analysis. This chapter presents the fundamental principles of constitutive laws together with their mathematical formulations and engineering applications in structural mechanics.

2. Generalized Hooke's Law

The generalized Hooke's law in three dimensions describes the relationship between the stress tensor and the strain tensor for isotropic linear elastic materials. Unlike the one-dimensional form, this formulation accounts for multidirectional stresses and deformations that occur in real engineering structures. It constitutes one of the fundamental equations of elasticity theory and is extensively used in structural mechanics, continuum mechanics, and finite element analysis.

2.1 Classical Hooke's Law

Classical Hooke's law is the fundamental constitutive relation used in linear elasticity. It establishes a proportional relationship between the normal stress applied to a material and the resulting normal strain, provided that the material remains within its elastic limit. This law is widely used in civil engineering to analyze the behavior of structural elements such as bars,

cables, columns, and reinforcement steel under axial loading. Hooke's law forms the basis for more advanced constitutive models used in solid mechanics and finite element analysis.

2.1.1 One -Dimensional State of Stress

In the one-dimensional case, Hooke's law describes the behavior of a bar subjected to axial tension or compression.

The relationship between stress and strain is expressed as:

$$\sigma = E\varepsilon$$

Where

- σ : normal stress (Pa or MPa)
- E : Young's modulus or modulus of elasticity
- ε : normal strain

➤ Normal stress is defined as the internal force distributed over the cross-sectional area:

$$\sigma = \frac{F}{A}$$

Where: F : axial force , a : cross-sectional area

➤ Normal strain measures the relative deformation of the material:

$$\varepsilon = \frac{\Delta L}{L_0}$$

Where: ΔL : change in length, L_0 : original length

➤ Young's modulus is a mechanical property that measures the stiffness of a material. It is defined by:

$$E = \frac{\sigma}{\varepsilon}$$

A material is said to behave linearly elastically when:

- Stress is proportional to strain.
- The material returns to its original shape after unloading.
- Deformations remain small.

The stress–strain curve in the linear elastic domain is a straight line whose slope equals Young's modulus.

Classical Hooke's law is widely applied in civil engineering for:

Steel bars in tension, reinforced concrete structures, bridge cables, truss members, columns under compression, deformation analysis.

2.2.2 Two-Dimensional State of Stress in Principal Axes

Let us consider a rectangular material element with a uniform thin thickness, subjected to a plane stress state characterized by the stresses σ_1 and σ_2 (Figure 4.2a).

Under the action of σ_1 alone, the rectangle elongates uniformly in direction 1 and contracts uniformly in the transverse direction 2 (Figure 4.2b). Therefore, it remains rectangular due to the isotropic nature of the material. The same behavior occurs under the action of σ_2 alone, or under the combined action of σ_1 and σ_2 according to the principle of superposition.

The element therefore undergoes no variation of the right angles, no shear distortion, and consequently the principal directions of stress (1 and 2) are also the principal directions of strain. This important property is generally no longer valid for anisotropic materials.

- (a) Action of σ_1 and σ_2
- (b) Deformed shape under σ_1 alone

- (c) Deformed shape under σ_2 alone

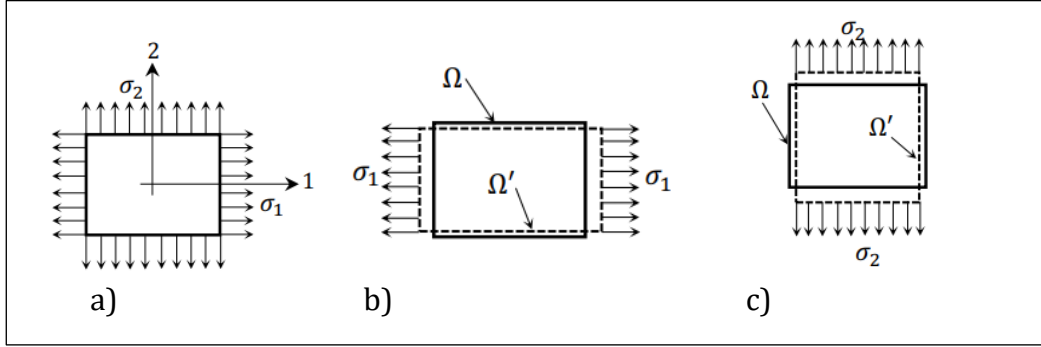


Figure 4.2 Plane stress state in principal axes

Let us now calculate the principal strain ε_1 .

Under the action of σ_1 alone, the following elongation appears (Fig. 1b):

$$\varepsilon'_1 = \frac{\sigma_1}{E}$$

Under the action of σ_2 alone, the lateral contraction phenomenon appears (Fig. 1c):

$$\varepsilon''_1 = \varepsilon_{lat} = -\nu \frac{\sigma_2}{E}$$

where ν is Poisson's ratio.

By superposition, when σ_1 and σ_2 act simultaneously:

$$\varepsilon_1 = \varepsilon'_1 + \varepsilon''_1 = \frac{\sigma_1 - \nu \sigma_2}{E}$$

Applying the same reasoning in direction 2 leads to a second equation of similar form. Together with the first one, they give the two-dimensional Hooke's law:

$$\varepsilon_1 = \frac{1}{E} (\sigma_1 - \nu \sigma_2), \quad \varepsilon_2 = \frac{1}{E} (\sigma_2 - \nu \sigma_1)$$

In matrix form

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{Bmatrix} = \frac{1}{E} \underbrace{\begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix}}_{\text{Compliance Matrix}} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix}$$

and inversely,

$$\begin{aligned} \sigma_1 &= \frac{E}{1 - \nu^2} (\varepsilon_1 + \nu \varepsilon_2) \\ \sigma_2 &= \frac{E}{1 - \nu^2} (\varepsilon_2 + \nu \varepsilon_1) \\ \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} &= \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{Bmatrix} \end{aligned}$$

The matrix above is called the stiffness matrix.

Remark : This law corresponds to the plane stress state and not to the plane strain state.

2.2.3 Three-Dimensional State in Principal Axes

By using a similar reasoning, the three-dimensional Hooke's law is obtained:

$$\begin{aligned} \varepsilon_1 &= \frac{1}{E} [\sigma_1 - \nu(\sigma_2 + \sigma_3)] \\ \varepsilon_2 &= \frac{1}{E} [\sigma_2 - \nu(\sigma_3 + \sigma_1)] \\ \varepsilon_3 &= \frac{1}{E} [\sigma_3 - \nu(\sigma_1 + \sigma_2)] \end{aligned}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix}$$

where the matrix is called the compliance matrix.

Or, by solving with respect to the stresses:

$$\begin{aligned} \sigma_1 &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_1 + \nu(\varepsilon_2 + \varepsilon_3)] \\ \sigma_2 &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_2 + \nu(\varepsilon_3 + \varepsilon_1)] \\ \sigma_3 &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_3 + \nu(\varepsilon_1 + \varepsilon_2)] \end{aligned}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu \\ \nu & (1-\nu) & \nu \\ \nu & \nu & (1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{Bmatrix}$$

The matrix above is called the stiffness matrix.

2.2 Generalized Hooke's Law in Three Dimensions

For isotropic materials, the constitutive law is written in compact tensor form as:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

Where $\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$, $\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \varepsilon_{yy} & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \varepsilon_{zz} \end{bmatrix}$

The symbol δ_{ij} is called the Kronecker delta,

The constants λ and μ are called Lamé elastic constants given by :

$$\begin{aligned} \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \\ \mu &= G = \frac{E}{2(1+\nu)} \end{aligned}$$

- The parameter λ is associated with volumetric deformation and compressibility of the material.
- The parameter μ is the shear modulus or modulus of rigidity, It measures the resistance of the material to shear deformation.

Physical Interpretation of the Equation

- First Term $\lambda \varepsilon_{kk} \delta_{ij}$ represents : volumetric deformation, hydrostatic effects, and uniform expansion or compression.
- Second Term $2\mu \varepsilon_{ij}$ represents: distortional deformation, shear effects, and shape change of the material.

Expanded Form of the Constitutive Equations

Normal Stress Components :

$$\begin{aligned} \sigma_{xx} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu \varepsilon_{xx} \\ \sigma_{yy} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu \varepsilon_{yy} \\ \sigma_{zz} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu \varepsilon_{zz} \end{aligned}$$

Shear Stress Components:

$$\tau_{xy} = 2\mu\epsilon_{xy}$$

$$\tau_{yz} = 2\mu\epsilon_{yz}$$

$$\tau_{xz} = 2\mu\epsilon_{xz}$$

2.3 Shear modulus

Consider the particular plane stress state called **pure shear**. It is characterized, on a rectangular element, by the principal stresses:

$$\sigma_1 = \sigma \text{ and } \sigma_2 = -\sigma$$

and, on a square element whose sides are oriented at 45° to the principal directions, only by tangential stresses of magnitude:

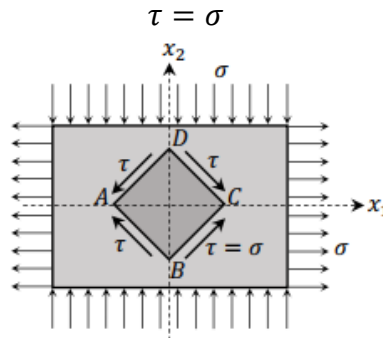


Figure 4.3 Pure Shear configuration.

Let us study the deformation of the square ABCD. Its sides do not change length since they are subjected to no normal stress (Figure 4.3). The diagonal AC elongates by the same amount that the diagonal BD shortens, due to the isotropy of the material. Moreover, introducing:

$$\sigma_1 = \sigma \text{ and } \sigma_2 = -\sigma$$

into relation (5), we obtain:

$$\epsilon_{AC} = -\epsilon_{BD} = \frac{\sigma(1 + \nu)}{E}$$

The deformed configuration of the square therefore becomes a rhombus: only the right angles change. Consequently, pure shear produces sliding deformation (Figure 4.4).

According to Hooke's law, there must be proportionality between the shear stress τ and the shear strain γ . By analogy with relation of classical Hooke's Law (1D), we write:

$$\tau = G\gamma, \quad \gamma = \frac{\tau}{G}$$

The elastic constant G , which has the dimension of stress (like E), is called the **shear modulus** or **Coulomb modulus**.

Where:

G : shear modulus, γ : shear strain (sliding deformation)

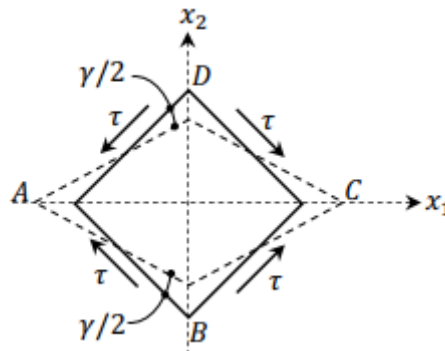


Figure 4.4 Deformation Due to Tangential Stresses.

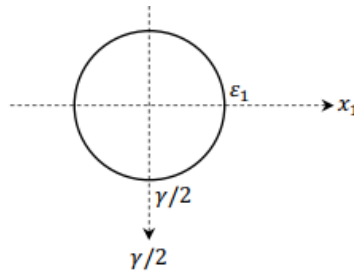


Figure 4 – Mohr's Circle of Strain in Pure Shear

However, since only two elastic constants are sufficient to characterize an isotropic Hookean solid, there must exist a relation linking E , ν , and G . Furthermore, the fact that a state of pure shear (Figure 4.4) can be extracted from a state where only normal stresses act (Figure 4.3) proves that such a relation must exist.

On the faces of square ABCD, only the half-shear strains $\gamma/2$ act (Figure 4.4), and the Mohr strain circle is centered at the origin (Figure 4.5). Along the principal direction x_1 , the normal strain is therefore:

$$\varepsilon_1 = \frac{\gamma}{2}$$

Moreover, in pure shear:

$$\sigma_1 = \tau \quad \text{and} \quad \sigma_2 = -\tau$$

Thus, Hooke's law may be written along direction x_1 :

$$\begin{aligned} \varepsilon_1 = \frac{\gamma}{2} &= \frac{1}{E}(\sigma_1 - \nu\sigma_2) \\ &= \frac{1}{E}(\tau + \nu\tau) = \frac{\tau(1 + \nu)}{E} \end{aligned}$$

From this relation, we obtain:

$$\tau = \frac{E}{2(1 + \nu)}\gamma$$

and by comparison with $\tau = G\gamma$, $\gamma = \frac{\tau}{G}$:

$$G = \frac{E}{2(1 + \nu)}$$

One important application of the shear modulus is the experimental determination of Poisson's ratio ν . The direct measurement of ν is delicate, whereas measuring G is relatively simple, particularly through a torsion test. Equation of G then allows accurate determination of ν when both E and G are experimentally measured.

2.4 Isotropic hooke's law in non-principal axes

Tangential stresses produce only shear deformations without volumetric expansion, while normal stresses produce only dilatations without shear deformation. Therefore, due to the isotropy of the material, there is no coupling between these two types of deformation.

2.4.1 Isotropic hooke's law

For each plane (x_i, x_j) with $(i \neq j)$, Hooke's law between the shear stress and the shear strain is:

$$\tau_{xy} = G\gamma_{xy} = G\gamma_{yz} = G\gamma_{xz}$$

and, in a three-dimensional Cartesian coordinate system (x,y,z) , the generalized Hooke's law is written as:

$$\begin{aligned}\varepsilon_{xx} &= \frac{1}{E} [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \\ \varepsilon_{yy} &= \frac{1}{E} [\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})] \\ \varepsilon_{zz} &= \frac{1}{E} [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \gamma_{yz} = \frac{\tau_{yz}}{G} \gamma_{xz} = \frac{\tau_{xz}}{G}\end{aligned}$$

The previous equations may be written in matrix form as:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}$$

This matrix is called the **compliance matrix**.

Using index notation, the six equations (12) may be written in the compact form:

$$\varepsilon_{ij} = \frac{1}{E} [(1 + \nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}]$$

where: δ_{ij} is the Kronecker delta, σ_{kk} is the trace of the stress tensor.

The volumetric strain is obtained by setting $i=j$ and summing:

$$\varepsilon_V = \varepsilon_{ii} = \frac{1 - 2\nu}{E} \sigma_{ii}$$

By writing: $\varepsilon_{ii} = \frac{1}{K} \sigma_0$, where σ_0 is the mean stress; the **Bulk Modulus** appears:

$$K = \frac{E}{3(1 - 2\nu)}$$

The bulk modulus characterizes the resistance of the material to volume variation.

2.4.2 Inverse Hooke's Law

The inverse constitutive relations are:

$$\begin{aligned}\sigma_{xx} &= \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz})] \\ \sigma_{yy} &= \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_{yy} + \nu(\varepsilon_{zz} + \varepsilon_{xx})] \\ \sigma_{zz} &= \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy})] \\ \tau_{xy} &= G\gamma_{xy} \tau_{yz} = G\gamma_{yz} \tau_{xz} = G\gamma_{xz}\end{aligned}$$

The matrix representation becomes:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1 - \nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1 - \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1 - 2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1 - 2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1 - 2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix}$$

This matrix is called the **rigidity matrix** or **stiffness matrix**.

In Index Form :

$$\sigma_{ij} = 2G(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij})$$

which can also be written as:

$$\sigma_{ij} = \frac{E}{(1+\nu)(1-2\nu)}[(1-2\nu)\varepsilon_{ij} + \nu\varepsilon_{kk}\delta_{ij}]$$

Instead of the engineering constants E, ν , and G, one may use the Lamé coefficients:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu \equiv G = \frac{E}{2(1+\nu)}$$

Then Hooke's law becomes:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

2.4.3 Deviatoric Form of Hooke's Law

Using the deviatoric stress tensor s_{ij} and the deviatoric strain tensor e_{ij} , Hooke's law may be written as:

$$s_{ij} = 2Ge_{ij}$$

and

$$\sigma_{kk} = 3K\varepsilon_{kk}$$

Where: s_{ij} : deviatoric stress tensor, e_{ij} : deviatoric strain tensor.

The equation s_{ij} describes shape distortion without volume change, The Equation σ_{kk} represents the volumetric deformation condition.

The table relating all these elastic constants is given below:

Table 4.2 Relations Between Elastic Constants

Known Constants	Relation	Expression
$(E, \nu) \rightarrow (\lambda)$	First Lamé coefficient	$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$
$(E, \nu) \rightarrow (\mu = G)$	Shear modulus (second Lamé coefficient)	$\mu = G = \frac{E}{2(1+\nu)}$
$(\lambda, \mu) \rightarrow (E)$	Young's modulus	$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$
$(\lambda, \mu) \rightarrow (\nu)$	Poisson's ratio	$\nu = \frac{\lambda}{2(\lambda + \mu)}$
$(\lambda, \mu) \rightarrow (K)$	Bulk modulus	$K = \lambda + \frac{2}{3}\mu$
$(K, G) \rightarrow (E)$	Young's modulus	$E = \frac{9KG}{3K + G}$
$(K, G) \rightarrow (\nu)$	Poisson's ratio	$\nu = \frac{3K - 2G}{2(3K + G)}$
$(E, G) \rightarrow (\nu)$	Poisson's ratio	$\nu = \frac{E}{2G} - 1$
$(E, \nu) \rightarrow (K)$	Bulk modulus	$K = \frac{E}{3(1-2\nu)}$
$(K, \nu) \rightarrow (E)$	Young's modulus	$E = 3K(1-2\nu)$

$(K, E) \rightarrow (\nu)$	Poisson's ratio	$\nu = \frac{1}{2} \left(1 - \frac{E}{3K} \right)$
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2.5 Hooke's Law in Orthotropic Material Case

An orthotropic material is a particular type of anisotropic material whose mechanical and physical properties are different along three mutually perpendicular directions called the principal material directions.

These three orthogonal directions are planes of material symmetry, meaning that the material behavior remains unchanged when observed with respect to these planes.

Typical examples of orthotropic materials include: Wood, Plywood, Fiber-reinforced composites Laminated materials, Rolled metal sheets.

The constitutive relation becomes:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix}$$

2.6 Hooke's Law in Anisotropic Material Case

An anisotropic material is a material whose mechanical properties depend on the direction considered. Unlike isotropic materials, the elastic behavior changes with orientation. Therefore:

- Stress and strain directions are not necessarily identical.
- The material may deform differently along different axes.
- Coupling between normal and shear components may exist.

The most general linear elastic constitutive law is:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$$

where: C_{ijkl} is fourth-order stiffness tensor

The inverse relation is:

$$\varepsilon_{ij} = S_{ijkl} \sigma_{kl}$$

Where: $S_{ijkl} = C_{ijkl}^{-1}$ is compliance tensor

Using Voigt notation, the Matrix Form of 3D constitutive law becomes:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}$$

The most general anisotropic material possesses 21 independent elastic constants.

3. Effect of temperature

Under the effect of heat, materials generally undergo thermal expansion. Let us consider a continuous medium subjected to a temperature field. When the temperature variation ΔT inside the material is continuous and uniform within the considered element, this variation mainly affects the strain tensor.

The relationship linking stresses, strains, and temperature is expressed by the following constitutive equation, in which the stress tensor depends linearly on both the strain tensor and the temperature variation:

$$\sigma = \sigma_0 + A:\varepsilon - K\theta$$

where:

- σ_0 : initial stress tensor
- A : fourth-order elasticity tensor
- $\theta = T - T_0$: temperature variation between the reference configuration at temperature T_0 and the current configuration at temperature T
- $K\theta$: thermal stresses generated by the temperature change

This formulation is valid for small temperature variations satisfying:

$$T - T_0 \ll T_0$$

3.1 Linear Isotropic Thermoelastic Constitutive Law

For a linear isotropic thermoelastic material, the previous relation simplifies into the classical thermoelastic law:

$$\sigma = \sigma_0 I + 2\mu\varepsilon + \lambda \text{tr}(\varepsilon)I - K\theta I$$

where:

- λ and μ : Lamé coefficients
- K : thermal coefficient
- $\sigma_0 I$: initial stress tensor
- $\text{tr}(\varepsilon)$: trace of the strain tensor
- I : identity tensor

The term $K\theta I$ represents the thermal stresses induced by temperature changes.

3.2 Inverse Thermoelastic Relation

Conversely, the strain tensor can be expressed as a function of stress and temperature variation:

$$\varepsilon = \varepsilon_0 I + \frac{1+\nu}{E}\sigma - \frac{\nu}{E}\text{tr}(\sigma)I + \alpha\theta I$$

where:

- E : Young's modulus
- ν : Poisson's ratio
- α : coefficient of thermal expansion
- ε_0 : initial strain tensor

The initial strain under zero stress and zero temperature variation is given by:

$$\varepsilon_0 = -\frac{(1-2\nu)}{E}\sigma_0$$

The thermal expansion coefficient is defined as:

$$\alpha = \frac{K}{3\lambda + 2\mu}$$

Physical Interpretation

- When temperature increases, most materials expand, producing additional strains called thermal strains.
- If thermal expansion is restrained, thermal stresses develop inside the structure.
- Thermoelasticity studies the coupling between mechanical deformation and thermal effects.

Common Civil Engineering Materials and Their Mechanical/Thermal Properties resumed in the table 4.2

Table 4.2 Common Materials and Their Mechanical/Thermal Properties.

Material	Young's Modulus (E) (GPa)	Poisson's Ratio (ν)	Coefficient of Thermal Expansion (α) (10 ⁻⁶ /°C)
Structural Steel	200 – 210	0.27 – 0.30	11 – 13
Stainless Steel	190 – 200	0.27 – 0.30	16 – 18
Aluminum	68 – 72	0.32 – 0.35	22 – 24
Cast Iron	100 – 170	0.21 – 0.30	9 – 12
Reinforced Concrete	25 – 40	0.15 – 0.25	10 – 14
Plain Concrete	20 – 35	0.15 – 0.20	9 – 12
High-Strength Concrete	40 – 55	0.18 – 0.25	10 – 13
Prestressed Concrete	35 – 50	0.18 – 0.22	10 – 12
Timber (Parallel to Grain)	8 – 16	0.25 – 0.40	3 – 6
Glass	60 – 80	0.20 – 0.27	8 – 10
Brick Masonry	5 – 20	0.15 – 0.25	5 – 8
Granite	40 – 70	0.20 – 0.30	7 – 9
Limestone	20 – 50	0.20 – 0.30	4 – 8
Asphalt Concrete	1 – 10	0.30 – 0.45	20 – 35
Copper	110 – 130	0.33 – 0.35	16 – 17
PVC	2 – 4	0.38 – 0.42	50 – 80

4. Strain energy

Strain energy is the internal energy stored within a material or structure when it deforms under the action of external forces. When a body is subjected to loading, stresses and strains develop inside the material, and the work done by the applied forces is transformed into elastic potential energy. As long as the deformation remains within the elastic range, this energy is recoverable when the loads are removed.

The concept of strain energy is fundamental in elasticity, structural mechanics, and continuum mechanics because it provides an energetic interpretation of deformation and mechanical behavior. It is widely used in the analysis of beams, plates, shells, machine components, aerospace structures, bridges, and advanced engineering materials. Strain energy methods are also essential in modern computational approaches such as the finite element method (FEM). For a simple linear elastic system subjected to a gradually applied force P producing a displacement δ , the strain energy stored in the body is:

$$U = \frac{1}{2}P\delta$$

This relation corresponds to the area under the load–displacement curve in the elastic domain.

4.1 Mathematical Formulation

In elasticity theory, strain energy may be expressed in terms of stress and strain components. For a uniaxial loading case, the strain energy stored in a body of volume V is given by:

$$U = \frac{1}{2} \sigma \varepsilon V$$

where:

- σ is the stress,
- ε is the strain,
- V is the volume of the material.

The strain energy per unit volume is called the strain energy density:

$$u = \frac{1}{2} \sigma \varepsilon$$

For a three-dimensional elastic body, the general tensorial form becomes:

$$u = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$$

and the total strain energy of the body is:

$$U = \frac{1}{2} \int_V \sigma_{ij} \varepsilon_{ij} dV$$

These expressions are fundamental in continuum mechanics and energy-based formulations.

4.2 Applications in Structural Mechanics

Different loading modes generate different forms of strain energy. For an axially loaded bar, the strain energy is expressed as:

$$U = \frac{P^2 L}{2AE}$$

where L is the length of the bar, A is the cross-sectional area, and E is Young's modulus.

In beam bending problems, the strain energy due to flexure is:

$$U = \int_0^L \frac{M^2(x)}{2EI} dx$$

where $M(x)$ is the bending moment and I is the moment of inertia.

For torsional loading, the strain energy becomes:

$$U = \int_0^L \frac{T^2(x)}{2GJ} dx$$

where $T(x)$ is the torque, G is the shear modulus, and J is the polar moment of inertia.

Strain energy methods are closely related to Castigliano's theorem:

$$\delta_i = \frac{\partial U}{\partial P_i}$$

which allows the calculation of displacements and rotations in structures.

Strain energy plays an essential role in structural analysis, fracture mechanics, vibration analysis, and finite element modeling. In fracture mechanics, regions with high strain energy density often correspond to crack initiation zones. In FEM, strain energy principles are used to derive stiffness matrices and numerical formulations for complex structures.

The concept is widely applied in civil engineering, aerospace engineering, renewable energy systems, composite materials, and biomechanics. Understanding how materials store and release energy helps engineers design safer, lighter, and more efficient structures.

Overall, strain energy provides a direct connection between forces, deformations, and energy storage, making it one of the most important concepts in elasticity and modern engineering mechanics.

5. Conclusion

This chapter presented the fundamental constitutive laws governing the mechanical behavior of deformable solids and established the relationship between stresses and strains in elastic materials. The generalized Hooke's law was developed progressively from one-dimensional loading conditions to two-dimensional and three-dimensional stress states, highlighting the importance of elasticity theory in continuum mechanics. Special attention was given to isotropic, orthotropic, and anisotropic materials, showing how material symmetry and directional properties influence mechanical response. The chapter also introduced the inverse and deviatoric forms of Hooke's law, as well as the role of the shear modulus and Lamé coefficients in elastic formulations. Thermoelastic effects were examined to demonstrate the influence of temperature variations on stress and deformation fields in materials and structures. Furthermore, the concept of strain energy was presented as a powerful energetic approach for analyzing deformation, structural behavior, and stability problems. The mathematical formulations and engineering applications discussed throughout this chapter constitute essential tools for structural analysis, finite element modeling, fracture mechanics, and advanced material design. Overall, constitutive laws provide the theoretical foundation necessary for understanding and predicting the mechanical behavior of engineering materials under various mechanical and thermal loading conditions.

Exercises

Exercise 1 One-Dimensional Hooke's Law

A steel bar of length $L=2\text{m}$ and cross-sectional area $A = 500\text{mm}^2$ is subjected to a tensile force $P=50\text{kN}$. The material has Young's modulus: $E=210\text{GPa}$, Poisson's ratio: $\nu=0.3$

Determine:

1. The normal stress.
2. The normal strain.
3. The elongation of the bar.
4. The lateral strain.
- 5.

Exercise 2 Two-Dimensional Plane Stress State

An isotropic elastic material is subjected to the following stresses:

$$\sigma_1 = 120 \text{ MPa}, \sigma_2 = 120 \text{ MPa}$$

The material properties are: $E=200 \text{ GPa}$, $\nu=0.25$.

Determine:

1. The strain ε_1
2. The strain ε_2

Exercise 3 — Three-Dimensional Hooke's Law

A material element is subjected to the following stresses:

$$\sigma_1 = 150 \text{ MPa}, \sigma_2 = 80 \text{ MPa}, \sigma_3 = 40 \text{ MPa}$$

The material properties are: $E=210 \text{ GPa}$, $\nu=0.3$

- Determine the strains: $\varepsilon_1, \varepsilon_2, \varepsilon_3$

Exercise 4 Thermoelasticity

A steel rod is fully restrained at both ends and subjected to a temperature increase:

$$\Delta T = 60^\circ \text{C}$$

Material properties: Young's modulus: $E=200 \text{ GPa}$, Thermal expansion coefficient: $\alpha = 12 \times 10^{-6} / ^\circ \text{C}$

Determine the thermal stress generated in the rod.

Exercise 5 Strain Energy in an Axially Loaded Bar

A steel bar has a Length $L=1.5 \text{ m}$, Cross-sectional area: $A = 400 \text{ mm}^2$, Young's modulus: $E=210 \text{ GPa}$

The bar is subjected to an axial tensile load: $P=80 \text{ kN}$

Determine the strain energy stored in the bar.

Exercise 6

Consider the following stress tensor:

$$\sigma = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (MPa)}$$

Knowing that $E=125 \text{ Mpa}$ and $\mu=45 \text{ Mpa}$, determine the components of the corresponding strain tensor.

Exercise 7

The following strain tensor is defined:

$$\varepsilon = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix} \times 10^{-2}$$

Deduce the stress tensor at point M, knowing that: $E = 21 \times 10^4 \text{ MPa}$, $\nu = 0.28$

Exercise 8

Consider a block composed of two cylindrical parts with axis (x-x), cross-sectional area S , and lengths l_1 and l_2 respectively. Both parts are made of the same linear isotropic thermoelastic material characterized by (E, ν, α) . These two parts are held between two fixed plates and separated by an expansion joint of thickness Δ .

Initially, the stresses are zero (natural state), the temperature is T_0 , and body forces are neglected. A uniform temperature increase is imposed: $\theta_1 = T_1 - T_0$

1. Determine the value of T_1 corresponding to the contact between the two parts.

Numerical data: $T_0 = 20^\circ \text{C}$, $E = 2.1 \times 10^5 \text{ Mpa}$, $\nu = 0.3$, $l_1 = 10 \text{ m}$, $l_2 = 7 \text{ m}$, $\Delta = 5 \text{ mm}$

2. Another temperature increase is then imposed: $\theta_2 = T_2 - T_1$. Calculate the resulting stress σ_{xx} in the two parts.

Numerical data: $\theta_2 = 10^\circ \text{C}$.

The constitutive relations are: $\sigma = \lambda (\text{tr } \varepsilon) \mathbf{I} + 2\mu \varepsilon - K\theta \mathbf{I}$, $\varepsilon = \frac{1+\nu}{E} \sigma - \frac{\nu}{E} (\text{tr } \sigma) \mathbf{I} + \alpha \theta \mathbf{I}$

Solutions

Solution to exercise 1

1. Calculate the normal stress

The normal stress is given by: $\sigma = \frac{P}{A}$

Convert the area:

$$A = 500 \text{ mm}^2 = 500 \times 10^{-6} \text{ m}^2$$

Convert the force:

$$P = 50 \text{ kN} = 50 \times 10^3 \text{ N}$$

Substituting:

$$\sigma = \frac{50 \times 10^3}{500 \times 10^{-6}} = 100 \text{ Mpa}$$

Step 2: Calculate the normal strain

Hooke's law in one dimension:

$$\varepsilon = \frac{\sigma}{E}$$

$$\varepsilon = \frac{100 \times 10^6}{210 \times 10^9} = 4.76 \times 10^{-4}$$

Step 3: Calculate the elongation

The elongation is:

$$\Delta L = \varepsilon L$$

$$\Delta L = 4.76 \times 10^{-4} \times 2 = 0.952 \text{ mm}$$

Step 4: Calculate the lateral strain

$$\varepsilon_{lat} = -\nu \varepsilon$$

$$\varepsilon_{lat} = -0.3 \times 4.76 \times 10^{-4} = -1.43 \times 10^{-4}$$

The negative sign indicates contraction.

Solution to exercise 2

For a plane stress state:

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu \sigma_2)$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu \sigma_1)$$

Calculate ε_1 and ε_2

Substitute the values:

$$\varepsilon_1 = \frac{1}{200 \times 10^9} (120 \times 10^6 - 0.25 \times 50 \times 10^6) = 5.375 \times 10^{-4}$$

$$\varepsilon_2 = \frac{1}{200 \times 10^9} (50 \times 10^6 - 0.25 \times 120 \times 10^6) = 1.0 \times 10^{-4}$$

Solution to exercise 3

The three-dimensional Hooke's law is:

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)]$$

$$\varepsilon_2 = \frac{1}{E}[\sigma_2 - \nu(\sigma_3 + \sigma_1)]$$

$$\varepsilon_3 = \frac{1}{E} [\sigma_3 - \nu(\sigma_1 + \sigma_2)]$$

$$\varepsilon_1 = \frac{1}{210 \times 10^9} [150 \times 10^6 - 0.3(80 \times 10^6 + 40 \times 10^6)] = 5.43 \times 10^{-4}$$

$$\varepsilon_2 = \frac{1}{210 \times 10^9} [80 \times 10^6 - 0.3(40 \times 10^6 + 150 \times 10^6)] = 1.1 \times 10^{-4}$$

$$\varepsilon_3 = \frac{1}{210 \times 10^9} [40 \times 10^6 - 0.3(150 \times 10^6 + 80 \times 10^6)] = -1.38 \times 10^{-4}$$

The negative sign indicates contraction.

Solution to exercise 4

For a fully restrained rod, the thermal stress is:

$$\sigma = E\alpha\Delta T$$

Substitute the values:

$$\sigma = 200 \times 10^9 \times 12 \times 10^{-6} \times 60 = 144 \text{ Mpa}$$

The stress is compressive because the expansion is prevented.

Solution to exercise 5

The strain energy for axial loading is:

$$U = \frac{P^2 L}{2AE}$$

Convert the quantities : $P = 80 \times 10^3 \text{ N}$, $A = 400 \times 10^{-6} \text{ m}^2$, $E = 210 \times 10^9 \text{ Pa}$

Substitute into the formula

$$U = \frac{(80 \times 10^3)^2 \times 1.5}{2 \times 400 \times 10^{-6} \times 210 \times 10^9} = 57.14 \text{ J}$$

This value represents the elastic energy stored in the bar during deformation.

Solution to exercise 6

Consider the following stress tensor:

$$\bar{\sigma} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\text{Mpa})$$

Determine the components of the strain tensor knowing that: $E = 125 \text{ MPa}$, $\mu = 45 \text{ MPa}$

The generalized Hooke's law is:

$$\varepsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \delta_{ij} \sigma_{kk}$$

The trace of the stress tensor is:

$$\sigma_{kk} = \text{tr} \left(\bar{\sigma} \right) = \sigma_{11} + \sigma_{22} + \sigma_{33} = 4 + 3 + 1 = 8 \text{ Mpa}$$

Using:

$$\mu = \frac{E}{2(1 + \nu)}$$

we obtain:

$$\nu = \frac{E}{2\mu} - 1 = \frac{125}{2 \times 45} - 1 = 0.39$$

Normal strain components

$$\begin{aligned}\varepsilon_{11} &= \frac{1+\nu}{E} \sigma_{11} - \frac{\nu}{E} \sigma_{kk} \\ \varepsilon_{11} &= \frac{1+0.39}{125} (4) - \frac{0.39}{125} (8) = 1.952 \times 10^{-2} \\ \varepsilon_{22} &= \frac{1+0.39}{125} (3) - \frac{0.39}{125} (8) = 0.84 \times 10^{-2} \\ \varepsilon_{33} &= \frac{1+0.39}{125} (1) - \frac{0.39}{125} (8) = -1.38 \times 10^{-2}\end{aligned}$$

Shear strain components

$$\varepsilon_{12} = \frac{1+\nu}{E} \sigma_{12} = \frac{1+0.39}{125} (2) = 2.22 \times 10^{-2}$$

because $\delta_{12} = 0$.

$\varepsilon_{13} = 0, \varepsilon_{23} = 0$

Therefore:

$$\varepsilon = \begin{bmatrix} 1.95 & 2.22 & 0 \\ 2.22 & 0.84 & 0 \\ 0 & 0 & -1.38 \end{bmatrix} \times 10^{-2}$$

Solution to exercise 7

The strain tensor at point M is:

$$\varepsilon = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix} \times 10^{-2}$$

Determine the stress tensor knowing that: $E = 21 \times 10^4$ Mpa, $\nu = 0.28$

The constitutive law is:

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$$

Lamé coefficients:

$$\begin{aligned}\mu &= \frac{E}{2(1+\nu)} = \frac{21 \times 10^4}{2(1+0.28)} = 8.2 \times 10^4 \\ \lambda &= \frac{\nu E}{(1-2\nu)(1+\nu)} = \frac{0.28 \times 21 \times 10^4}{(1-2 \times 0.28)(1+0.28)} = 10.44 \times 10^4\end{aligned}$$

Components of the stress tensor

For $i=1, j=1$

$$\sigma_{11} = (\lambda + 2\mu) \varepsilon_{11} + \lambda \varepsilon_{22} + \lambda \varepsilon_{33}$$

Since $\varepsilon_{11} = 0$:

$$\sigma_{11} = \lambda(\varepsilon_{22} + \varepsilon_{33}) = 4 \times 10^{-2} \times 10.44 \times 10^4 = 41.76 \times 10^2 \text{ Mpa}$$

For $i=1, j=2$

$$\sigma_{12} = 2\mu \varepsilon_{12} = 0$$

For $i=1, j=3$

$$\sigma_{13} = 2\mu \varepsilon_{13} = 2 \times 8.2 \times 10^4 \times (-1) \times 10^{-2} = -16.4 \times 10^2 \text{ Mpa}$$

For $i=2, j=2$

$$\sigma_{22} = \lambda \varepsilon_{11} + (\lambda + 2\mu) \varepsilon_{22} + \lambda \varepsilon_{33}$$

$$\sigma_{22} = 2 \times 10^{-2} (10.44 \times 10^4 + 2 \times 8.2 \times 10^4) + 10.44 \times 10^4 \times 2 \times 10^{-2} = 74.56 \times 10^2 \text{ MPa}$$

For $i=2, j=3$

$$\sigma_{23} = 2\mu \varepsilon_{23} = 2 \times 8.2 \times 10^4 \times 2 \times 10^{-2} = 32.8 \times 10^2 \text{ Mpa}$$

For $i=3, j=3$

$$\sigma_{33} = \lambda \varepsilon_{11} + \lambda \varepsilon_{22} + (\lambda + 2\mu) \varepsilon_{33}$$

$$\sigma_{33} = 2 \times 10^{-2} (2 \times 10.44 \times 10^4 + 2 \times 8.2 \times 10^4) = 74.56 \times 10^2 \text{ MPa}$$

therefore:

$$\bar{\sigma} = \begin{bmatrix} 41.76 & 0 & -16.4 \\ 0 & 74.56 & 32.8 \\ -16.4 & 32.8 & 74.56 \end{bmatrix} \times 10^2 \text{ (MPa)}$$

Solution to exercise 8

1. Determination of T_1 corresponding to contact between the two parts

We consider a uniaxial stress state:

$$\bar{\sigma} = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The strain tensor is:

$$\bar{\varepsilon} = \frac{1 + \nu}{E} \bar{\sigma} - \frac{\nu}{E} \left(\text{tr} \bar{\sigma} \right) \bar{I} + \alpha \theta \bar{I}$$

Thus:

$$\bar{\varepsilon} = \begin{bmatrix} \frac{\sigma_{xx}}{E} + \alpha \theta & 0 & 0 \\ 0 & -\frac{\nu \sigma_{xx}}{E} + \alpha \theta & 0 \\ 0 & 0 & -\frac{\nu \sigma_{xx}}{E} + \alpha \theta \end{bmatrix}$$

The expansion joint allows free expansion without stress:

$$\Delta \neq 0 \Rightarrow \sigma_{xx} = 0$$

Hence:

$$\varepsilon_{xx} = \alpha \theta_1$$

For part 1:

$$\Delta l_1 = \varepsilon_{xx} l_1$$

For part 2:

$$\Delta l_2 = \varepsilon_{xx} l_2$$

Contact occurs when:

$$\Delta l_2 = \varepsilon_{xx} l_2$$

Therefore:

$$\varepsilon_{xx} (l_1 + l_2) = \Delta$$

$$\alpha \theta_1 (l_1 + l_2) = \Delta$$

$$\alpha (T_1 - T_0) (l_1 + l_2) = \Delta$$

Hence:

$$T_1 = \frac{\Delta}{\alpha (l_1 + l_2)} + T_0$$

Substituting the numerical values:

$$T_1 = \frac{5 \times 10^{-3}}{12 \times 10^{-6} (10 + 7)} + 20$$

2. Stress resulting from an additional temperature increase θ_2

After contact, the two parts can no longer elongate:

$$\varepsilon_{xx} = 0$$

Thus:

$$\frac{\sigma_{xx}}{E} + \alpha\theta_2 = 0$$

Therefore:

$$\sigma_{xx} = -E\alpha\theta_2$$

Substituting:

$$\sigma_{xx} = -2.1 \times 10^5 \times 12 \times 10^{-6} \times 10 = -25.2 \text{ Ma (Compression stress.)}$$

Chapter 5: General Equations of Linear Elasticity

- 1. Introduction**
- 2. Navier–Cauchy equation**
- 3. Beltrami–Michell equations**
- 4. Saint-Venant Principle**
- 5. Conclusion**

1. Introduction

Linear elasticity constitutes one of the fundamental pillars of continuum mechanics and solid mechanics. It provides the mathematical framework used to describe the behavior of deformable solids subjected to external loads while remaining within the elastic domain.

The term “linear” refers to the proportional relationship between stresses and strains, commonly expressed through Hooke’s law. This assumption is valid when deformations are sufficiently small and when the material behavior remains elastic.

The theory of elasticity generally assumes that materials are homogeneous and isotropic. A homogeneous material possesses identical properties at every point, whereas an isotropic material exhibits identical mechanical behavior in all directions. Although real engineering materials may not always perfectly satisfy these assumptions, they provide sufficiently accurate approximations for many practical applications.

The unknown quantities in elasticity problems may be expressed either in terms of displacements or stresses. Depending on the chosen formulation, different systems of differential equations can be established. Among the most important formulations are:

- Lamé equations, expressed in terms of displacement components.
- Beltrami–Michell equations, expressed directly in terms of stresses.

These equations constitute the general equations of linear elasticity and form the basis for analytical methods, numerical techniques, and finite element formulations.

The formulation of elasticity problems also requires the specification of boundary conditions. Two principal categories exist:

- Displacement boundary conditions, where the displacements are prescribed.
- Traction boundary conditions, where surface forces are specified.

A complete elasticity problem consists of:

- The governing differential equations
- Constitutive relations
- Compatibility conditions
- Appropriate boundary conditions

In practice, solving elasticity equations exactly is often difficult because of the complexity of geometries and loading conditions. Consequently, approximate numerical techniques such as the finite element method (FEM) have become essential tools in modern engineering analysis. The present chapter introduces the fundamental equations governing linear elasticity. Particular attention is devoted to Lamé equations, Beltrami–Michell equations, and Saint-Venant’s principle. These concepts constitute the theoretical basis for advanced methods used in structural analysis and computational mechanics.

The chapter also emphasizes the physical interpretation of elasticity equations and their engineering significance. Understanding the derivation and application of these equations allows engineers to model structural behavior accurately and to design safer and more efficient systems.

2. Navier–Cauchy equation

2.1 General Concept

The governing displacement equations of isotropic linear elasticity are commonly called the Navier–Cauchy equations; they involve the Lamé parameters λ and μ . These equations describe the equilibrium of an elastic body in terms of displacement components rather than stresses or strains.

This displacement-based form is especially useful because once the displacement field is known, strains and stresses can be obtained directly from the kinematic and constitutive relations. For a three-dimensional Cartesian system, the displacement vector is

$$u = (u, v, w)$$

Where u , v and w are the displacement components in the x , y , and z directions, respectively.

The derivation is based on:

- Equilibrium equations.
- Strain–displacement relations.
- Hooke’s law for isotropic linear elastic materials.

By eliminating stresses and strains in favor of displacements, one obtains a system of second-order partial differential equations.

2.2 Equilibrium Equations

For a body subjected to body forces (f_x, f_y, f_z) , the static equilibrium equations are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0$$

These equations express the balance of internal stresses and external body forces in an infinitesimal material element.

2.3 Strain–Displacement Relations

For small deformations, the normal strains are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

The engineering shear strains are

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

These relations connect the displacement field to the deformation field. They are valid under the assumptions of small strains and small rotations.

2.4 Constitutive Relations

For isotropic linear elastic materials, Hooke's law can be written in terms of the Lamé constants as

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$$

The Lamé constants are related to Young's modulus E and Poisson's ratio ν by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

These parameters characterize the elastic response of isotropic materials.

2.5 Derivation of the Governing Equations

Substituting the strain–displacement relations into Hooke's law and then into the equilibrium equations yields the displacement form of linear elasticity:

$$(\lambda + \mu) \nabla(\nabla \cdot u) + \mu \nabla^2 u + f = 0$$

where:

- $\nabla \cdot u$ is the divergence of the displacement field.
- $\nabla^2 u$ is the vector Laplacian.
- f is the body force vector per unit volume.

This is the **standard form of the Navier–Cauchy equations** for static isotropic elasticity.

The expanded cartesian form of this equations become

$$(\lambda + \mu) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \mu \nabla^2 u + f_x = 0$$

$$(\lambda + \mu) \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \mu \nabla^2 v + f_y = 0$$

$$(\lambda + \mu) \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \mu \nabla^2 w + f_z = 0$$

These equations govern the displacement field inside an elastic solid.

Physical Interpretation

The term involving $\nabla(\nabla \cdot \mathbf{u})$ is associated with volumetric deformation, while the term $\nabla^2 \mathbf{u}$ is associated with distortional or shape-changing deformation. Together, they describe how internal elastic forces balance external body forces.

The equations also show coupling between the displacement components. A displacement in one direction can influence deformation in other directions through Poisson effects.

Special Cases

a) Plane Stress : For thin plates subjected to in-plane loading, the stress normal to the thickness is negligible: $\sigma_{zz} = 0$

This reduces the 3D elasticity problem to an effective two-dimensional formulation.

b) Plane Strain : For long structures where deformation along one direction is negligible:

$$\varepsilon_{zz} = 0$$

This assumption is commonly used in geotechnical engineering, tunnel analysis, and dam or retaining wall modeling.

c) Axisymmetric Problems

When the geometry and loading are symmetric about an axis, the governing equations can be written in cylindrical coordinates, which simplifies the analysis.

2.6 Engineering Applications

The Navier–Cauchy equations are widely used in:

- Structural mechanics.
- Pressure vessel analysis.
- Tunnel engineering.
- Geomechanics.
- Finite element analysis.
- Aerospace structures.
- Mechanical component design.

Their displacement-based form makes them especially suitable for numerical methods such as the finite element method.

Modern finite element methods are largely built on displacement formulations derived from linear elasticity. In most structural simulations, nodal displacements are the primary unknowns, and stresses and strains are recovered afterward.

These equations also form the theoretical basis for elastic wave propagation, vibration analysis, and dynamic behavior of structures.

3. Beltrami–Michell equations

The Beltrami–Michell equations are an important formulation in linear elasticity, written directly in terms of stress components rather than displacements. They are especially useful when the stress field is the primary unknown and when one wants to enforce both equilibrium and compatibility in a stress-based formulation.

These equations were developed through the work of Eugenio Beltrami and John Henry Michell, who made major contributions to classical elasticity theory.

In many engineering problems, stress is the key quantity because failure criteria are usually expressed in terms of stress limits. Typical examples include:

- Yielding of metals.
- Cracking of concrete.
- Fracture mechanics.
- Fatigue analysis.
- Pressure vessel design.

A stress formulation can be useful in analytical elasticity because it gives direct access to stress distributions without first solving for displacements.

3.1 Compatibility Conditions

Compatibility conditions ensure that the strain field corresponds to a continuous displacement field. In three-dimensional elasticity, they are derived from the strain–displacement relations. One of the classical compatibility equations is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

This relation is correct when γ_{xy} is the engineering shear strain. More generally, the full three-dimensional compatibility system contains six independent equations. If compatibility is not satisfied, the strain field may be mathematically possible but physically impossible because it cannot come from a single-valued displacement field.

3.4 Derivation of the Equations

The Beltrami–Michell equations are obtained by combining:

- equilibrium equations,
- Hooke's law,
- strain compatibility conditions.

For homogeneous isotropic linear elastic materials without body forces, a standard form is lamé equation is

$$(\lambda + G) \frac{\partial \varepsilon_v}{\partial x} + G \nabla^2 u + F_x = 0$$

$$(\lambda + G) \frac{\partial \varepsilon_v}{\partial y} + G \nabla^2 v + F_y = 0$$

$$(\lambda + G) \frac{\partial \varepsilon_v}{\partial z} + G \nabla^2 w + F_z = 0$$

where:

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Under the effect of temperature variations, the equations become:

$$(\lambda + G) \frac{\partial \varepsilon_v}{\partial x} + G \nabla^2 u - \frac{\alpha E}{1 - 2\nu} \frac{\partial (\Delta T)}{\partial x} + F_x = 0$$

$$(\lambda + G) \frac{\partial \varepsilon_v}{\partial y} + G \nabla^2 v - \frac{\alpha E}{1 - 2\nu} \frac{\partial (\Delta T)}{\partial y} + F_y = 0$$

$$(\lambda + G) \frac{\partial \varepsilon_v}{\partial z} + G \nabla^2 w - \frac{\alpha E}{1 - 2\nu} \frac{\partial (\Delta T)}{\partial z} + F_z = 0$$

By differentiation of Lamé equations respectively with respect to x, y, and z, assuming that the body forces are constant ($F_i = \text{constant}$).

$$\begin{aligned}(\lambda + G) \frac{\partial^2 \varepsilon_v}{\partial x^2} + G \frac{\partial}{\partial x} (\nabla^2 u) &= 0 \\(\lambda + G) \frac{\partial^2 \varepsilon_v}{\partial y^2} + G \frac{\partial}{\partial y} (\nabla^2 v) &= 0 \\(\lambda + G) \frac{\partial^2 \varepsilon_v}{\partial z^2} + G \frac{\partial}{\partial z} (\nabla^2 w) &= 0\end{aligned}\tag{5.1}$$

By adding those equations term by term, we obtain:

When the body forces are constant, the Laplacian of the volumetric strain is zero.

$$\begin{aligned}(\lambda + G) \left(\frac{\partial^2 \varepsilon_v}{\partial x^2} + \frac{\partial^2 \varepsilon_v}{\partial y^2} + \frac{\partial^2 \varepsilon_v}{\partial z^2} \right) + G \nabla^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) &= 0 \\(\lambda + 2G) \nabla^2 \varepsilon_v &= 0; \quad \nabla^2 \varepsilon_v = 0\end{aligned}$$

It is therefore necessary to transform the strains in equations into stresses. For this purpose, Lamé's equations are used. Applying the Laplacian operator to these equations gives:

$$\begin{aligned}\{\nabla^2(\sigma_{xx}) = \lambda \nabla^2(\varepsilon_v) + 2G \nabla^2(\varepsilon_{xx}) \quad \nabla^2(\sigma_{yy}) = \lambda \nabla^2(\varepsilon_v) + 2G \nabla^2(\varepsilon_{yy}) \quad \nabla^2(\sigma_{zz}) \\= \lambda \nabla^2(\varepsilon_v) + 2G \nabla^2(\varepsilon_{zz}) \Rightarrow \{\nabla^2(\sigma_{xx}) = 2G \nabla^2(\varepsilon_{xx}) \quad \nabla^2(\sigma_{yy}) \\= 2G \nabla^2(\varepsilon_{yy}) \quad \nabla^2(\sigma_{zz}) = 2G \nabla^2(\varepsilon_{zz})\end{aligned}\tag{5.2}$$

According to the generalized Hooke's law, we have:

$$\theta = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = (3\lambda + 2G)\varepsilon_v \Rightarrow \varepsilon_v = \frac{\theta}{3\lambda + 2G}$$

Differentiating this previous equation twice with respect to x, y, and z, we obtain:

$$\begin{aligned}\frac{\partial^2 \varepsilon_v}{\partial x^2} &= \frac{1}{(3\lambda + 2G)} \frac{\partial^2 \theta}{\partial x^2} \\ \frac{\partial^2 \varepsilon_v}{\partial y^2} &= \frac{1}{(3\lambda + 2G)} \frac{\partial^2 \theta}{\partial y^2} \\ \frac{\partial^2 \varepsilon_v}{\partial z^2} &= \frac{1}{(3\lambda + 2G)} \frac{\partial^2 \theta}{\partial z^2}\end{aligned}\tag{5.3}$$

We must now substitute equations (5.3) and (5.2) into equations (5.1). This yields the Beltrami–Michell equations:

$$\begin{aligned}(1 + \nu) \nabla^2(\sigma_{xx}) + \frac{\partial^2 \theta}{\partial x^2} &= 0; \quad (1 + \nu) \nabla^2(\tau_{xy}) + \frac{\partial^2 \theta}{\partial x \partial y} = 0 \\(1 + \nu) \nabla^2(\sigma_{yy}) + \frac{\partial^2 \theta}{\partial y^2} &= 0; \quad (1 + \nu) \nabla^2(\tau_{xz}) + \frac{\partial^2 \theta}{\partial x \partial z} = 0 \\(1 + \nu) \nabla^2(\sigma_{zz}) + \frac{\partial^2 \theta}{\partial z^2} &= 0; \quad (1 + \nu) \nabla^2(\tau_{yz}) + \frac{\partial^2 \theta}{\partial y \partial z} = 0\end{aligned}$$

5. Saint-Venant Principle

Saint-Venant's principle is a principle of linear elasticity which states that the effects of the initial distribution of stresses over a small region of a material gradually vanish as one moves away from that region. In other words, at a sufficiently large distance from the load application zone, the elastic behavior of a material is essentially determined by the boundary conditions rather than by local details.

Saint-Venant compatibility equations are a set of equations that describe the compatibility of strains in an elastic material. These equations are named after the French mathematician and engineer Adhémar Jean Claude Barré de Saint-Venant, who formulated these relations.

In the mathematical theory of elasticity, the Saint-Venant compatibility condition defines the relation between the strain tensor ε and a displacement field u by:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad 1 \leq i, j \leq 3$$

For second-order tensor fields, the Saint-Venant compatibility equations written in index notation become:

$$\frac{\partial^2}{\partial x_k \partial x_k} \varepsilon_{ij} + \frac{\partial^2}{\partial x_i \partial x_j} \varepsilon_{kk} - \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_k} \varepsilon_{ik} \right) - \frac{\partial^2}{\partial x_i \partial x_k} \varepsilon_{jk} = 0, \quad 1 \leq i, j \leq 3$$

Where: $\varepsilon_{kk} = \text{trace}[\varepsilon]$,

- The repeated index k implies summation (Einstein summation convention).
- The two non-repeated indices i and j each vary from 1 to 3.
-

4. Conclusion

The general equations of linear elasticity constitute the mathematical foundation for analyzing deformable solids subjected to mechanical loading. These equations combine equilibrium conditions, constitutive relations, and compatibility requirements to provide a complete description of stress, strain, and displacement fields within elastic materials.

Among the various formulations, Lamé equations occupy a central position because they express elasticity problems directly in terms of displacement components. Their displacement-based formulation is especially important in computational mechanics and finite element analysis, where nodal displacements are the primary unknowns.

The Beltrami–Michell equations provide an alternative stress-based formulation of elasticity. These equations are particularly useful in analytical elasticity and stress concentration problems, where direct knowledge of stress fields is essential. Their relation to Airy stress functions and biharmonic equations has enabled the development of many classical elasticity solutions.

Saint-Venant's principle introduces an essential engineering approximation that simplifies complex loading conditions. By recognizing that localized stress effects diminish with distance, engineers can replace complicated load distributions with equivalent simplified systems. This principle forms the basis of many structural theories and numerical approaches.

Exercises

Exercise 1

The three displacement components U , V , and W at a point P of an elastic solid are given by:

$$\left\{ U = \frac{5}{4}kx - \frac{1}{3}ky \quad V = \frac{1}{4}kx - \frac{\sqrt{7}}{4}ky \quad W = 0 \right.$$

where k is a non-zero constant

The solid is assumed to have an isotropic linear elastic behavior characterized by: $E = 210000 \text{ MPa}$; $\nu = 0.25$

1. Determine the strain tensor $[\epsilon]$.
2. Determine the stress tensor $[\sigma]$.

Exercise 2 Lamé Equations

A homogeneous and isotropic elastic body is subjected to body forces neglected in all directions. The displacement field is given by:

$$u = ax^2, v = by^2, w = cz^2$$

where a, b, and c are constants.

For an elastic material characterized by Lamé constants λ and μ :

1. Determine the strain components.
2. Determine the stress components using Hooke's law.
3. Verify whether the displacement field satisfies the Lamé equations in the absence of body forces.

Exercise 2 Beltrami-Michell Equations

Consider a two-dimensional stress field in an elastic plate:

$$\sigma_x = 6y, \sigma_y = 4x, \tau_{xy} = 2x + 3y$$

Verify whether this stress field satisfies the Beltrami-Michell compatibility equation.

Exercise 3 — Saint-Venant Principle

A prismatic steel bar of cross-sectional area: $A = 500 \text{ mm}^2$ is subjected to an axial tensile force: $P = 50 \text{ kN}$

Two loading cases are considered:

Case 1: uniformly distributed load over the entire cross section,

Case 2: concentrated load applied over a small region of the cross section.

Using Saint-Venant's principle:

1. Determine the normal stress far from the loaded end.
2. Explain the difference in stress distribution near and far from the load application region.

Solutions

Solution to exercise 1

1) Determination of the strain tensor

The strain tensor can be expressed in matrix form as:

$$[\epsilon(P)]_{P, (i,j) \in I} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} & \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} & \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) & \text{Sym} & \frac{\partial v}{\partial y} & \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

Thus:

$$[\varepsilon] = \begin{bmatrix} \frac{5}{4}k & \frac{1}{2}(-\frac{1}{12}k) & 0 & \frac{1}{2}(-\frac{1}{12}k) & -\frac{\sqrt{7}}{4}k & 0 & 0 & 0 & 0 \\ \frac{1}{2}(-\frac{1}{12}k) & \frac{5}{4}k & 0 & \frac{1}{2}(-\frac{1}{12}k) & -\frac{\sqrt{7}}{4}k & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{4}k & 0 & -\frac{\sqrt{7}}{4}k & 0 & 0 & 0 & 0 \\ \frac{1}{2}(-\frac{1}{12}k) & \frac{1}{2}(-\frac{1}{12}k) & 0 & \frac{5}{4}k & -\frac{\sqrt{7}}{4}k & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{7}}{4}k & -\frac{\sqrt{7}}{4}k & 0 & -\frac{\sqrt{7}}{4}k & \frac{5}{4}k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{4}k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4}k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4}k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4}k \end{bmatrix}$$

2) Determination of the stress tensor

The stress tensor is obtained from the constitutive relation:

$$[\sigma(M)] = \lambda \operatorname{tr}[\varepsilon(M)][I] + 2\mu[\varepsilon(M)]$$

where the Lamé coefficients are:

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}; \mu = G = \frac{E}{2(1+\nu)}$$

Finally:

$$[\sigma] = 21000k \begin{bmatrix} (15 - \sqrt{7}) & -\frac{1}{3} & 0 & -\frac{1}{3} & (5 - 3\sqrt{7}) & 0 & 0 & 0 & (5 - \sqrt{7}) \\ -\frac{1}{3} & (15 - \sqrt{7}) & 0 & -\frac{1}{3} & (5 - 3\sqrt{7}) & 0 & 0 & 0 & (5 - \sqrt{7}) \\ 0 & 0 & (15 - \sqrt{7}) & -\frac{1}{3} & (5 - 3\sqrt{7}) & 0 & 0 & 0 & (5 - \sqrt{7}) \\ -\frac{1}{3} & -\frac{1}{3} & 0 & (15 - \sqrt{7}) & (5 - 3\sqrt{7}) & 0 & 0 & 0 & (5 - \sqrt{7}) \\ (5 - 3\sqrt{7}) & (5 - 3\sqrt{7}) & 0 & (5 - 3\sqrt{7}) & (15 - \sqrt{7}) & 0 & 0 & 0 & (5 - \sqrt{7}) \\ 0 & 0 & 0 & 0 & 0 & (15 - \sqrt{7}) & -\frac{1}{3} & 0 & (5 - \sqrt{7}) \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & (15 - \sqrt{7}) & -\frac{1}{3} & (5 - \sqrt{7}) \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & (15 - \sqrt{7}) & (5 - \sqrt{7}) \\ (5 - \sqrt{7}) & (5 - \sqrt{7}) & (5 - \sqrt{7}) & (5 - \sqrt{7}) & (5 - \sqrt{7}) & (5 - \sqrt{7}) & (5 - \sqrt{7}) & (5 - \sqrt{7}) & (15 - \sqrt{7}) \end{bmatrix}$$

Solution to exercise 2

1) Determination of Strains

The normal strain components are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

Thus

$$\varepsilon_{xx} = 2ax, \quad \varepsilon_{zz} = 2cz, \quad \varepsilon_{yy} = 2by$$

The shear strains are:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \quad \gamma_{yz} = 0, \quad \gamma_{zx} = 0$$

$$\varepsilon = [2ax \ 0 \ 0 \ 0 \ 2by \ 0 \ 0 \ 0 \ 2cz]$$

2) Determination of Stresses

Hooke's law for isotropic elasticity is:

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$$

The volumetric strain is:

$$\varepsilon_{kk} = 2ax + 2by + 2cz$$

Thus:

$$\sigma_{xx} = \lambda(2ax + 2by + 2cz) + 4\mu ax$$

$$\sigma_{yy} = \lambda(2ax + 2by + 2cz) + 4\mu by$$

$$\sigma_{zz} = \lambda(2ax + 2by + 2cz) + 4\mu cz$$

The shear stresses are zero: $\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$

3) Verification of Lamé Equations

The Lamé equations without body forces are:

$$(\lambda + \mu)\nabla(\nabla \cdot u) + \mu\nabla^2 u = 0$$

$$\nabla \cdot u = 2ax + 2by + 2cz$$

Thus: $\nabla(\nabla \cdot u) = [2a \ 2b \ 2c]$

The Laplacian of the displacement vector is: $\nabla^2 u = [2a \ 2b \ 2c]$

Substituting into Lamé equations: $(\lambda + \mu)[2a \ 2b \ 2c] + \mu[2a \ 2b \ 2c] = (\lambda + 2\mu)[2a \ 2b \ 2c] = 0$

Therefore $a = b = c = 0$

Hence, the given displacement field satisfies Lamé equations only for the trivial case.

Solution to exercise 3

$$\sigma_x = 6y, \sigma_y = 4x, \tau_{xy} = 2x + 3y$$

For plane elasticity, the compatibility equation is:

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} = 2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

Compute the Second Derivatives

$$\frac{\partial \sigma_x}{\partial y} = 6; \quad \frac{\partial^2 \sigma_x}{\partial y^2} = 0$$

$$\frac{\partial \sigma_y}{\partial x} = 4, \quad \frac{\partial^2 \sigma_y}{\partial x^2} = 0$$

$$2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0$$

The stress field satisfies the Beltrami–Michell compatibility equation.

Chapter 6: Solution of Plane Elasticity Problems

1. Introduction
2. Plane strain problems
3. Plane stress problems
4. Fundamental Equations of Plane Elasticity
5. Airy stress function
6. Conclusion

1. Introduction

Plane elasticity problems are a class of two-dimensional elasticity problems in which the stress and strain fields depend only on two spatial coordinates, usually x and y . They arise when the geometry, loading, and deformation of a body can be idealized as planar, which makes the analysis much simpler than the full three-dimensional case. In classical elasticity, the two main plane formulations are plane stress and plane strain.

These problems are important because many practical engineering structures can be modeled accurately using 2D elasticity, such as thin plates, long tunnels, retaining structures, and parts with geometric symmetry. A major advantage of plane elasticity is that it allows the use of the Airy stress function, which reduces the governing equations to a single biharmonic equation under appropriate assumptions. This makes the chapter a central bridge between classical analytical elasticity and modern computational mechanics.

In plane elasticity, the main objective is to determine the stress field and, when needed, the corresponding displacements and strains. The choice between plane stress and plane strain depends on the thickness, geometry, and boundary conditions of the body. These idealizations are not exact for every structure, but they are widely accepted because they often capture the dominant mechanical response with good accuracy.

2. Plane strain problems

If we consider a case in which the displacement w is zero and the displacements u and v are independent of z , it follows that the strain components are defined by:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad 1 \leq i, j \leq 3$$

and:

$$\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

The strain components ε_{xx} , ε_{yy} and ε_{xy} are independent of z , which defines a state of plane strain.

According to Hooke's law:

$$\varepsilon_{zz} = \frac{1}{E} [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]$$

Since the material is not free to deform along the z -direction, the stress component is given by:

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

Therefore, the strain and stress tensors reduce to:

$$[\varepsilon] = [\varepsilon_{xx} \ \varepsilon_{xy} \ 0 \ \varepsilon_{xy} \ \varepsilon_{yy} \ 0 \ 0 \ 0 \ 0], \quad [\sigma] = [\sigma_{xx} \ \sigma_{xy} \ 0 \ \sigma_{xy} \ \sigma_{yy} \ 0 \ 0 \ 0 \ \sigma_{zz}]$$

We observe that the strain state is biaxial (plane strain), whereas the stress state is triaxial, meaning that it contains three non-zero principal stresses.

By replacing σ_{zz} with its value $\nu(\sigma_{xx} + \sigma_{yy})$, it is nevertheless possible to write a two-dimensional form of Hooke's law

$$\{\varepsilon_{xx} = \frac{1}{E'} (\sigma_{xx} - \nu' \sigma_{yy}) \ \varepsilon_{yy} = \frac{1}{E'} (\sigma_{yy} - \nu' \sigma_{xx}) \ \varepsilon_{xy} = \frac{\sigma_{xy}}{2\mu}$$

with the effective elastic moduli: $E' = \frac{E}{1-\nu^2}$, $\nu' = \frac{\nu}{1-\nu}$

3. Plane stress problems

A plane stress state can be defined by the following conditions:

$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$$

or equivalently, the stress components σ_{xx} , σ_{yy} and σ_{xy} are independent of z .

According to Hooke's law:

$$\{\varepsilon_{xx} = \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) \ \varepsilon_{yy} = \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) \ \varepsilon_{xy} = \frac{\sigma_{xy}}{2\mu}$$

Considering the third coordinate z , we obtain:

$$\varepsilon_{zz} = -\frac{\nu}{E} (\sigma_{xx} + \sigma_{yy}), \ \varepsilon_{xz} = 0, \ \varepsilon_{yz} = 0$$

Therefore, the stress and strain tensors reduce to:

$$[\sigma] = [\sigma_{xx} \ \sigma_{xy} \ 0 \ \sigma_{xy} \ \sigma_{yy} \ 0 \ 0 \ 0 \ 0], \quad [\varepsilon] = [\varepsilon_{xx} \ \varepsilon_{xy} \ 0 \ \varepsilon_{xy} \ \varepsilon_{yy} \ 0 \ 0 \ 0 \ \varepsilon_{zz}]$$

This is a biaxial stress state (with two non-zero principal stresses), whereas the strain state is triaxial.

4. Fundamental Equations of Plane Elasticity

The fundamental equations of plane elasticity (plane strain problems and plane stress problems) may be grouped as follows.

4.1 Equations of Static Analysis

a) Equilibrium Equations

In the absence of body forces, the equilibrium equations are:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

b) Boundary Conditions

$$\{X = \sigma_{xx}l + \sigma_{xy}m \quad Y = \sigma_{xy}l + \sigma_{yy}m$$

Where $n = (l, m)$ is the outward unit normal vector to the boundary, making an angle θ_n with the x-axis.

4.2 Equations of Geometrical Analysis

Cauchy Strain Equations (Extensions and Shear Distortions) are :

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

And Saint-Venant Compatibility Equation is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

So the equations of Physical Analysis are

- Hooke's Equations mentioned previously the plan stress or strain problems

Plane Strain	Plane Stress
$\begin{cases} \varepsilon_{xx} = \frac{1}{E'}(\sigma_{xx} - \nu'\sigma_{yy}) \\ \varepsilon_{yy} = \frac{1}{E'}(\sigma_{yy} - \nu'\sigma_{xx}) \\ \varepsilon_{xy} = \frac{\sigma_{xy}}{2\mu} \end{cases}$	$\begin{cases} \varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \\ \varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) \\ \varepsilon_{xy} = \frac{\sigma_{xy}}{2\mu} \end{cases}$

- Lamé Equations

Plane Strain	Plane Stress
$\begin{cases} \sigma_{xx} = (\lambda + 2G)\varepsilon_{xx} + \lambda\varepsilon_{yy} \\ \sigma_{yy} = (\lambda + 2G)\varepsilon_{yy} + \lambda\varepsilon_{xx} \\ \sigma_{xy} = G\gamma_{xy} \\ \varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} \end{cases}$	$\begin{cases} \sigma_{xx} = 2G\varepsilon_{xx} + \lambda\varepsilon_v \\ \sigma_{yy} = 2G\varepsilon_{yy} + \lambda\varepsilon_v \\ \sigma_{xy} = G\gamma_{xy} \\ \varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \end{cases}$

5. Airy stress function

Several methods exist for solving plane elasticity problems, each adapted to specific geometries and boundary conditions. For simple geometries and well-defined boundary conditions, analytical solutions may sometimes be obtained using Airy stress functions.

For both plane strain and plane stress conditions, the same two-dimensional Hooke's law of plane strain can be written, formally analogous to the three-dimensional law, provided that effective elastic moduli are used.

For plane strain: $E' = \frac{E}{1-\nu^2}$, $\nu' = \frac{\nu}{1-\nu}$

For plane stress: $E' = E$, $\nu' = \nu$

In both cases, elastic equilibrium in the absence of body forces ($\text{div } \sigma = 0$) is written as:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

These equations are automatically satisfied if one can find a scalar function $\Phi(x, y)$, called the Airy stress function (Airy, 1862), such that:

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

For plane problems, the compatibility condition may be written in terms of stresses (Maxwell, 1868) as:

$$\Delta(\sigma_{xx} + \sigma_{yy}) = 0$$

In terms of the Airy stress function, solving a plane elasticity problem reduces to finding a biharmonic function satisfying the boundary conditions:

$$\Delta \Delta \Phi(x, y) = 0$$

or equivalently:

$$\frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

The boundary conditions are given by Equation :

$$\{X = \sigma_{xx}l + \sigma_{xy}m \quad Y = \sigma_{xy}l + \sigma_{yy}m$$

It should be noted that when the boundary conditions are expressed in terms of stresses, both plane states admit the same stress-field solution. The strain and displacement fields differ only by the effective elastic moduli.

Thus, any biharmonic function satisfies the equilibrium equations, Hooke's law, and the compatibility equation. Under the assumption of small deformations, the solution to an elasticity problem is unique. Consequently, the Airy function Φ satisfying the boundary conditions provides the unique solution of the problem.

We now consider the more general case where body forces F derive from a potential V such that:

$$F = \nabla V, \quad X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}$$

The equilibrium equations then become:

$$\left\{ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial V}{\partial x} = 0 \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial V}{\partial y} = 0 \right.$$

These equations are satisfied if the stresses derive from an Airy function Φ :

$$\left\{ \sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} + V \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} + V \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \right.$$

The stress potential function must then satisfy a biharmonic equation.

For plane strain: $\Delta \Delta \Phi = -\frac{1-\nu}{1-2\nu} \Delta V$

For plane stress: $\Delta \Delta \Phi = (1-\nu) \Delta V$

5. Conclusion

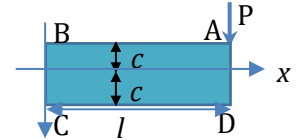
In this chapter, the fundamental methods used for solving plane elasticity problems were presented and analyzed. The concepts of plane strain and plane stress were introduced to

model two-dimensional elastic behavior under different physical assumptions. The governing equations of elasticity, including equilibrium, compatibility, and constitutive relations, were established as the basis of the analysis. The equations of static and geometrical analysis were also discussed to describe stress, strain, and displacement fields in elastic solids. Particular attention was given to the Airy stress function, which provides an elegant and powerful method for satisfying the equilibrium equations automatically. Finally, these formulations constitute essential tools in solid mechanics and engineering applications for the analytical treatment of structural and mechanical problems.

Exercises

Exercise 1

Consider a cantilever beam with a narrow rectangular cross-section (whose width is taken as unity), bent by a force P applied at its free end . The self-weight is neglected.



Demonstrate that the function $\Phi(x, y) = \frac{Pxy}{4c^3}(3c^2 - y^2)$ is an Airy stress function.

Solutions

Solution to exercise 1

For $\Phi(x, y)$ to be an Airy stress function, three conditions must be satisfied.

1. The Function Must Be Biharmonic

$$\Delta\Delta\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = 0$$

$$\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

After differentiation, we obtain:

$$\frac{\partial^4\Phi}{\partial x^4} = 0, \quad \frac{\partial^4\Phi}{\partial x^2\partial y^2} = 0, \quad \frac{\partial^4\Phi}{\partial y^4} = 0$$

Hence: $\Delta\Delta\Phi = 0$ Therefore, the function is biharmonic.

2. The Function Must Generate a Stress Distribution

Using the Airy relations:

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}$$

we obtain:

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2} = -\frac{3Px}{2c^2} + \frac{3Pxy}{2c^3} = \frac{3Px}{2c^3}(y - c)$$

$$\sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} = 0$$

$$\sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} = \frac{3P}{2c^2}y - \frac{3P}{4c^3}y^2 = \frac{3P}{4c^3}(2cy - y^2)$$

Thus, the function generates a valid stress field.

3.The Stress Distribution Must Satisfy the Boundary Conditions

$$\begin{aligned} \{X = \sigma_{xx}l + \sigma_{xy}m = \left(-\frac{3P}{2c^2}x + \frac{3P}{2c^3}xy\right)l + \left(\frac{3P}{2c^2}y - \frac{3P}{4c^3}y^2\right)m \quad Y = \sigma_{xy}l + \sigma_{yy}m \\ = \left(\frac{3P}{2c^2}y - \frac{3P}{4c^3}y^2\right)l \end{aligned}$$

- For free faces AB and CD: $\vec{n}(0, -1), \vec{n}(0, 1) \setminus$ and $y = -c + c$
 $X = 0, Y = 0$ This is an unloaded surface.
- For faces AD: $\vec{n}(1, 0); x=l$

$$\{X = 0 \quad Y = -\frac{3P}{2c^2}y + \frac{3P}{4c^3}y^2$$

The face is loaded only along the y-direction by the distributed force P.

The resultant force is:

$$\begin{aligned} \int_0^{2c} Y dy &= \int_0^{2c} \left(-\frac{3P}{2c^2}y + \frac{3P}{4c^3}y^2\right) dy \\ &= \left[-\frac{3P}{2c^2} \frac{y^2}{2} + \frac{3P}{4c^3} \frac{y^3}{3}\right]_0^{2c} \\ &= -3P + 2P = -P \end{aligned}$$

Displacement Boundary Conditions

Face BC (Fixed End) $n(-1, 0), x = 0, -c < y < c$

$$\{X = -\frac{3P}{2c^2}l + \frac{3P}{2c^3}ly \quad Y = \frac{3P}{2c^2}y - \frac{3P}{4c^3}y^2$$

We now calculate the reactions along the x- and y-directions and the moment at the fixed support.

Reaction Along the x-Direction

$$\int_0^{2c} X dy = \int_0^{2c} \left(-\frac{3Pl}{2c^2} + \frac{3Ply}{2c^3}\right) dy = \left[-\frac{3Pl}{2c^2}y + \frac{3Pl}{2c^3} \frac{y^2}{2}\right]_0^{2c} = 0$$

$$R_x = 0$$

$$\int_0^{2c} Y dy = \int_0^{2c} \left(-\frac{3P}{2c^2}y + \frac{3P}{4c^3}y^2\right) dy = \left[-\frac{3P}{2c^2} \frac{y^2}{2} + \frac{3P}{4c^3} \frac{y^3}{3}\right]_0^{2c} = -3P + 4P = P$$

$$R_y = P$$

The moment at the fixed support is:

$$\int_0^{2c} Xy dy = \int_0^{2c} \left(-\frac{3Pl}{2c^2} + \frac{3Ply}{2c^3}\right) y dy = \left[-\frac{3Pl}{2c^2} \frac{y^2}{2} + \frac{3Pl}{2c^3} \frac{y^3}{3}\right]_0^{2c} = -3Pl + 4Pl = Pl$$

Hence, the moment at the fixed end is: $M = Pl$

Therefore, $\Phi(x, y)$ is an Airy stress function.

Chapter 7: Beam Bending

1. Introduction
2. Euler–Bernoulli Beam Theory
3. Timoshenko Beam Theory
4. Conclusion

1. Introduction

Beam bending is one of the most important topics in the theory of elasticity and structural mechanics. Beams are structural elements designed to support transverse loads, which produce internal stresses and deformations. The analysis of beam bending allows engineers to determine stresses, deflections, and stability conditions in structures such as bridges, buildings, machine components, and aircraft structures.

The study of beam bending is based on simplifying assumptions that make it possible to model the mechanical behavior of slender structural elements. Among the most important beam theories are the Euler–Bernoulli beam theory and the Timoshenko beam theory, which are widely used in engineering analysis and design.

2. Definitions

A beam is a structural element designed to support loads acting perpendicular to its longitudinal axis. It is characterized by a length that is much greater than its cross-sectional dimensions, which allows it to resist bending effects efficiently. The centroid of the beam sections forms a reference curve called the neutral axis or mean line of the beam.

Beams are widely used in engineering structures because they can transmit loads safely from one point to another while limiting excessive deformation.

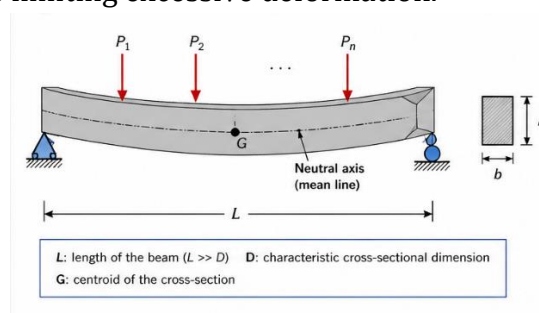


Figure 7.1 schematic representation of a beam.

Main Characteristics of Beams

- The beam length is generally much larger than its width or height: $L \gg D$
- Beams are subjected mainly to transverse loads.
- Internal stresses appear due to bending moments and shear forces.
- The geometry of the cross-section strongly influences the beam resistance and stiffness.

Some frequently used beam sections are: Rectangular section, Square section, Circular section; Hollow circular section, I-section (steel beams), T-section and channel section

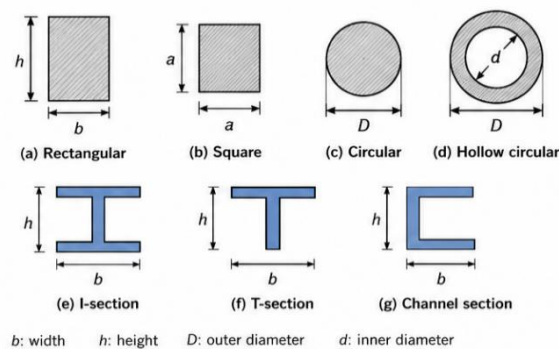


Figure 7.2: Examples of common beam cross-section

3. Euler–Bernoulli Beam Theory

The Euler–Bernoulli beam theory, also called the classical beam theory, is one of the earliest and most widely used models for beam bending analysis. This theory assumes that plane cross-sections remain plane and perpendicular to the neutral axis after deformation. It also neglects the effects of shear deformation, making it particularly suitable for long and slender beams.

The theory establishes a direct relationship between the bending moment and the curvature of the beam. It provides accurate results for small deflections and is extensively applied in civil and mechanical engineering. The Euler–Bernoulli model forms the foundation of many structural analysis methods and remains essential in the study of elasticity and strength of materials.

3.1 Fundamental Assumptions

The Euler–Bernoulli beam theory, also known as the classical beam theory, is one of the most important models used in structural mechanics and elasticity. This theory assumes that transverse shear deformations are negligible compared with longitudinal deformations. It is mainly applicable to long and slender beams where bending effects dominate the structural response.

The theory is based on the following assumptions:

- Plane cross-sections remain plane after deformation.
- Cross-sections remain perpendicular to the neutral axis during bending.
- The material is homogeneous, isotropic, and linearly elastic.
- Deflections and rotations are small.
- Shear deformation effects are neglected.

Because of these simplifications, the Euler–Bernoulli model provides an efficient and accurate approximation for many engineering structures such as bridges, building beams, and machine elements.

3.2 Governing Equation of Beam Bending

The fundamental differential equation governing the deflection of an Euler–Bernoulli beam is:

$$EI \frac{d^2 w}{dx^2} = -M(x)$$

where:

- $w(x)$: vertical displacement (deflection) of the beam,
- E : Young's modulus of the material,
- I : second moment of area of the cross-section,
- $M(x)$: bending moment distribution along the beam.

This equation establishes the relationship between the applied bending moment and the curvature of the elastic line of the beam.

Stress Components

According to Euler–Bernoulli theory, the dominant stress component is the normal bending stress:

$$\sigma_{xx} = -Ey \frac{d^2 w}{dx^2}$$

while the transverse normal stress and shear stress are neglected:

$$\begin{cases} \sigma_{yy} = 0 \\ \tau_{xy} = 0 \end{cases}$$

Strain Components

The corresponding strain field is given by:

$$\varepsilon_{xx} = -y \frac{d^2 w}{dx^2}$$

with:

$$\begin{cases} \varepsilon_{yy} = 0 \\ \gamma_{xy} = 0 \end{cases}$$

These relations indicate that longitudinal strains vary linearly through the beam thickness, while transverse and shear strains are neglected.

4. Timoshenko Beam Theory

The Timoshenko beam theory was developed to improve the limitations of the Euler–Bernoulli model, especially for short and thick beams. Unlike the classical theory, Timoshenko theory takes into account the effects of shear deformation and rotational inertia. This leads to more accurate predictions of beam deflections and stresses when the beam geometry or loading conditions make shear effects significant.

In this theory, cross-sections remain plane after deformation but are not necessarily perpendicular to the neutral axis. The inclusion of shear deformation makes the Timoshenko model more suitable for modern engineering applications involving composite materials, sandwich structures, and dynamic analysis. Consequently, it is widely used in advanced structural mechanics and finite element modeling.

4.1 Fundamental Assumptions

The Timoshenko beam theory was developed to overcome the limitations of the Euler-Bernoulli model, particularly for short and thick beams where shear deformation becomes significant. Unlike the classical theory, Timoshenko theory incorporates both transverse shear effects and rotational deformation of the cross-sections.

The principal assumptions of this theory are:

- Cross-sections remain plane after deformation.
- Cross-sections are not necessarily perpendicular to the neutral axis.
- Shear deformation is included in the analysis.
- Rotational effects of the beam sections are considered.
- The material behavior remains linear elastic.

This theory is especially suitable for beams subjected to heavy transverse loads, sandwich structures, composite beams, and dynamic structural analysis.

4.2 Governing Equation

The governing equation of the Timoshenko beam model can be expressed as:

$$EI \frac{d^2 w}{dx^2} + GJ \frac{d^2}{dx^2} \left(\frac{dw}{dx} \right) = -M(x)$$

where:

- G: shear modulus of the material,
- J: polar moment of inertia of the section,
- the remaining quantities retain their previous definitions.

This equation reflects the contribution of both bending and shear deformation to the beam behavior.

Stress Components

In Timoshenko beam theory, both normal and shear stresses are taken into account:

$$\begin{cases} \sigma_{xx} = -Ey \frac{d^2 w}{dx^2} \\ \tau_{xy} = -G \frac{d^2 w}{dx^2} \end{cases}$$

The transverse normal stress may also appear depending on loading conditions and beam geometry.

Strain Components

The deformation components are expressed as:

$$\begin{cases} \varepsilon_{xx} = -y \frac{d^2 w}{dx^2} \\ \gamma_{xy} = -\frac{d^2 w}{dx^2} \end{cases}$$

These relations demonstrate that the Timoshenko model accounts for both bending strains and shear strains, providing a more realistic description of beam deformation.

Stress Components:

$$\sigma_{xx} = -E \frac{d^2 w}{dx^2}, \quad \sigma_{yy} = -\frac{G}{2} \frac{d^2 v}{dx^2}, \quad \tau_{xy} = -G \frac{d^2 w}{dx^2}$$

Strain Components:

$$\varepsilon_{xx} = -\frac{d^2 w}{dx^2}, \quad \varepsilon_{yy} = -\frac{1}{2} \frac{d^2 v}{dx^2}, \quad \gamma_{xy} = -\frac{d^2 u}{dx^2}$$

5. Conclusion

Beam bending theory plays a fundamental role in elasticity and structural engineering because it provides the tools required to analyze the behavior of structural elements subjected to transverse loads. The Euler–Bernoulli beam theory offers a simple and efficient model for slender beams where shear deformation can be neglected. In contrast, the Timoshenko beam theory introduces the effects of shear deformation and rotational inertia, allowing a more realistic representation of thick and short beams. Together, these theories constitute the basis of modern beam analysis and are essential for the safe and efficient design of engineering structures.

Exercises

Exercise 1

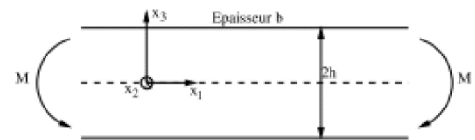
Consider a beam fixed at one end and simply supported at the other end, subjected to a concentrated load at its midpoint.

Calculate the normal stresses and normal strains at the midspan section of the beam.

Exercise 2

The beam has a rectangular cross-section of height $2h$ and width b . It is subjected to pure bending (shear neglected), and it is assumed that a cross-section normal to axis x_1 remains plane. The material behavior is elastic (E, ν).

Determine the stress and strain distributions in elasticity.



Solutions

Solution to exercise 1

The vertical displacement at the middle of the beam is:

$$v = \frac{PL^3}{48EI}$$

The normal stresses σ_{xx} and normal strains ε_{xx} can be calculated from bending according to Euler–Bernoulli beam theory:

$$\sigma_{xx} = -\frac{My}{I}, \quad \varepsilon_{xx} = -\frac{y}{R}$$

The bending moment M at the middle of the beam due to the concentrated load is:

$$M = \frac{PL}{4}$$

The radius of curvature R is related to the bending moment by:

$$R = \frac{EI}{M}$$

Solution to exercise 2

In pure bending, the stress state is uniaxial. The displacement u_1 , therefore ε_{xx} and σ_{xx} , are proportional to x_3 (origin of axes at the neutral axis).

The stress and strain tensors become:

$$[\sigma] = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, [\varepsilon] = \begin{pmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & -\frac{\nu\sigma}{E} & 0 \\ 0 & 0 & -\frac{\nu\sigma}{E} \end{pmatrix}$$

The moment equilibrium equation is:

$$M = \int_{-h}^{+h} x_3 \sigma_{xx} b \, dx_3$$

Assuming: $\sigma_{xx} = kx_3$

$$\sigma_{xx} = \sigma = M \frac{x_3}{I}, I = \frac{2bh^3}{3}$$

The maximum stress is:

$$\sigma_{\max} = \sigma_m = \frac{3M}{2bh^2}, \quad \sigma(x_3) = \left(\frac{x_3}{h}\right) \sigma_m$$

Chapter 8: Study of Thin Plates

- 1. Introduction**
- 2. Fundamental Assumptions**
- 3. Kinematics of a Thin Plate**
- 4. Deformation of a Thin Plate**
- 5. Stress Tensor and Constitutive Law**
- 6. Equilibrium Equations of a Plate**
- 7. Conclusion**

1. Introduction

Thin plates are structural elements widely used in engineering because of their high strength-to-weight ratio and their ability to resist transverse loads. A plate is considered thin when its thickness is very small compared to its other dimensions, such as length and width. Thin plate theory is essential in structural mechanics, aerospace engineering, shipbuilding, civil engineering, and mechanical design.

The study of thin plates aims to determine the displacement, stresses, strains, and stability of plate structures subjected to different loading conditions. Classical thin plate theory, also known as Kirchhoff-Love plate theory, provides a simplified mathematical model suitable for small deflections and elastic behavior.

Applications of thin plates include floors, bridge decks, aircraft wings, machine components, and storage tanks. Understanding the behavior of thin plates allows engineers to design lighter, safer, and more efficient structures.

A plate is a flat structural element whose thickness is very small compared to its length and width. The middle plane of the plate is called the mid-surface.

Depending on the ratio between thickness and other dimensions, plates may be classified as:

- Thin plates
- Moderately thick plates
- Thick plates

A Cartesian coordinate system is generally used:

- x and y : coordinates in the plane of the plate
- z : coordinate through the thickness

The displacement of the plate mainly occurs in the transverse direction and is denoted by $w(x, y)$. Thin plates differ from beams because bending occurs in two directions instead of one.

2. Fundamental Assumptions

The classical theory of thin plates is based on several simplifying assumptions that reduce the complexity of the three-dimensional elasticity problem.

1. **Small Thickness Assumption** : The thickness h of the plate is much smaller than its in-plane dimensions: $h \ll a, h \ll b$, where a and b are the plate dimensions.
2. **Small Deflection Assumption** : The transverse deflection w is small compared to the plate thickness. Therefore, geometric nonlinearities can be neglected.
3. **Normality Assumption** : Straight lines normal to the mid-surface before deformation remain straight and normal after deformation.
4. **Negligible Transverse Shear Deformation**: The transverse shear strains are assumed negligible: $\gamma_{xz} = \gamma_{yz} = 0$
5. **Linear Elastic Material** : The material obeys Hooke's law and behaves elastically within the working range.

These assumptions form the basis of Kirchhoff plate theory.

We also consider the following two principles:

- a) **Saint-Venant's principle**: The behavior at any point in the material is independent of the manner in which the forces are applied (provided that this point is sufficiently far from the point of application), and can therefore be modeled by a resultant system of internal forces at that point.
- b) **The principle of superposition**: The order in which external loads are applied to the body is not important; that is, regardless of the order of application of the loads, we obtain the same final state. This principle makes it possible to decompose a complex loading into a sum of elementary loadings (tension, compression, shear, torsion, bending).

3. Kinematics of a Thin Plate

The kinematics of thin plates describes the displacement field and motion of points inside the plate during deformation.

According to Kirchhoff theory, the displacement field is (Eq. 8.1):

$$u(x, y, z) = -z \frac{\partial w}{\partial x}, \quad v(x, y, z) = -z \frac{\partial w}{\partial y}, \quad w = w(x, y)$$

where:

- u and v are in-plane displacements
- w is the transverse displacement

The rotations of the normal are (Eq. 8.2):

$$\theta_x = -\frac{\partial w}{\partial x}, \quad \theta_y = -\frac{\partial w}{\partial y}$$

The displacement field shows that in-plane displacements vary linearly through the thickness.

4. Deformation of a Thin Plate

We can consider the linearized Green–Lagrange strain tensor to model this deformation as a function of displacement.

This tensor is symmetric, and its components written using Voigt notation are defined by (Eq. 8.3):

$$\varepsilon^g = \begin{pmatrix} \varepsilon_{xx}^g \\ \varepsilon_{yy}^g \\ \varepsilon_{zz}^g \\ \varepsilon_{yz}^g \\ \varepsilon_{xz}^g \\ \varepsilon_{xy}^g \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial X_1} \\ \frac{\partial u_2}{\partial X_2} \\ \frac{\partial u_3}{\partial X_3} \\ \frac{1}{2} \left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \right) \end{pmatrix} = \begin{pmatrix} \frac{\partial u(x, y, z)}{\partial x} \\ \frac{\partial v(x, y, z)}{\partial y} \\ \frac{\partial w(x, y, z)}{\partial z} \\ \frac{1}{2} \left(\frac{\partial v(x, y, z)}{\partial z} + \frac{\partial w(x, y, z)}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial u(x, y, z)}{\partial z} + \frac{\partial w(x, y, z)}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial u(x, y, z)}{\partial y} + \frac{\partial v(x, y, z)}{\partial x} \right) \end{pmatrix}$$

The strain tensor are (Eq. 8.4) :

$$\left\{ \begin{array}{l} \varepsilon_{xx}^g = \frac{\partial u_0(x, y)}{\partial x} + z \frac{\partial \theta_y(x, y)}{\partial x} \\ \varepsilon_{yy}^g = \frac{\partial v_0(x, y)}{\partial y} - z \frac{\partial \theta_x(x, y)}{\partial y} \\ \varepsilon_{zz}^g = \frac{\partial w_0(x, y)}{\partial z} \\ \varepsilon_{yz}^g = \frac{1}{2} \left(\frac{\partial v_0(x, y)}{\partial z} - \theta_x(x, y) + \frac{\partial w_0(x, y)}{\partial y} \right) \\ \varepsilon_{xz}^g = \frac{1}{2} \left(\frac{\partial u_0(x, y)}{\partial z} + \theta_y(x, y) + \frac{\partial w_0(x, y)}{\partial x} \right) \\ \varepsilon_{xy}^g = \frac{1}{2} \left[\frac{\partial u_0(x, y)}{\partial y} + \frac{\partial v_0(x, y)}{\partial x} + z \left(\frac{\partial \theta_y(x, y)}{\partial y} - \frac{\partial \theta_x(x, y)}{\partial x} \right) \right] \end{array} \right.$$

The strain tensor in Voigt notation (Eq. 8.5)

$$\varepsilon^g = \begin{pmatrix} \varepsilon_{xx}^g \\ \varepsilon_{yy}^g \\ \varepsilon_{zz}^g \\ \varepsilon_{yz}^g \\ \varepsilon_{xz}^g \\ \varepsilon_{xy}^g \end{pmatrix} = \begin{pmatrix} \frac{\partial u_0(x, y)}{\partial x} + z \frac{\partial \theta_y(x, y)}{\partial x} \\ \frac{\partial v_0(x, y)}{\partial y} + z \left(-\frac{\partial \theta_x(x, y)}{\partial y} \right) \\ 0 \\ \frac{1}{2} \left(\frac{\partial w_0(x, y)}{\partial y} - \theta_x(x, y) \right) = 0 \\ \frac{1}{2} \left(\frac{\partial w_0(x, y)}{\partial x} + \theta_y(x, y) \right) = 0 \\ \frac{1}{2} \left(\frac{\partial u_0(x, y)}{\partial y} + \frac{\partial v_0(x, y)}{\partial x} \right) + z \frac{1}{2} \left(\frac{\partial \theta_y(x, y)}{\partial y} - \frac{\partial \theta_x(x, y)}{\partial x} \right) \end{pmatrix}$$

We can decompose this global strain tensor to reveal:

- The membrane strain tensor (Eq. 8.6):

$$\varepsilon_m = \begin{pmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_0(x, y)}{\partial x} \\ \frac{\partial v_0(x, y)}{\partial y} \\ \frac{1}{2} \left(\frac{\partial u_0(x, y)}{\partial y} + \frac{\partial v_0(x, y)}{\partial x} \right) \end{pmatrix}$$

- The bending strain tensor (Eq. 8.7):

$$\varepsilon_b = \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_y(x, y)}{\partial x} \\ -\frac{\partial \theta_x(x, y)}{\partial y} \\ \frac{1}{2} \left(\frac{\partial \theta_y(x, y)}{\partial y} - \frac{\partial \theta_x(x, y)}{\partial x} \right) \end{pmatrix}$$

- The shear strain tensor (Eq. 8.8):

$$\varepsilon_c = \begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix} = \begin{pmatrix} \frac{\partial w_0(x, y)}{\partial x} + \theta_y(x, y) \\ \frac{\partial w_0(x, y)}{\partial y} - \theta_x(x, y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The deformation tensor for the thin plate — modeling both membrane deformations (in-plane translation (x, y)) and bending (rotation about the x and y axes) — is defined as follows : (Eq. 8.9):

$$\varepsilon^g = \begin{pmatrix} \varepsilon_{xx}^g \\ \varepsilon_{yy}^g \\ \varepsilon_{xy}^g \end{pmatrix} = \begin{pmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{pmatrix} + z \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{pmatrix} = \varepsilon_m + z \varepsilon_b$$

Or (Eq. 8.10)

$$\left\{ \begin{array}{l} \varepsilon_{xx}^g = \frac{\partial u_0(x, y)}{\partial x} + z \frac{\partial \theta_y(x, y)}{\partial x} \\ \varepsilon_{yy}^g = \frac{\partial v_0(x, y)}{\partial y} - z \frac{\partial \theta_x(x, y)}{\partial y} \\ \varepsilon_{xy}^g = \frac{1}{2} \left(\frac{\partial u_0(x, y)}{\partial y} + \frac{\partial v_0(x, y)}{\partial x} \right) + \frac{1}{2} z \left(\frac{\partial \theta_y(x, y)}{\partial y} - \frac{\partial \theta_x(x, y)}{\partial x} \right) \end{array} \right.$$

5. Stress Tensor and Constitutive Law

The stress tensor for a thin plate is expressed as follows (Eq. 8.11):

$$\sigma = \begin{pmatrix} \sigma_{xx}(x, y) & \sigma_{xy}(x, y) & \sigma_{xz} = 0 \\ \sigma_{yx}(x, y) & \sigma_{yy}(x, y) & \sigma_{yz} = 0 \\ \sigma_{zx} = 0 & \sigma_{zy} = 0 & \sigma_{zz} = 0 \end{pmatrix}, \sigma = \begin{pmatrix} \sigma_{xx}(x, y) \\ \sigma_{yy}(x, y) \\ \sigma_{xy}(x, y) \end{pmatrix}$$

According to Hooke's law expressed as a function of E and ν , the relation between the stress tensor σ and the strain tensor ε takes the following matrix form (Eq. 8.12):

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{1 - \nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 - \nu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}$$

Considering the linearized Green–Lagrange strain tensor established in equation (8.9), the constitutive law for the thin plate becomes (Eq. 8.13):

$$\begin{cases} \sigma_{xx} = \frac{E}{1-\nu^2} (\varepsilon_{xx} + \nu \varepsilon_{yy} + \nu \kappa_{xx} + \nu \kappa_{yy}) \\ \sigma_{yy} = \frac{E}{1-\nu^2} (\nu \varepsilon_{xx} + \varepsilon_{yy} + \nu \kappa_{xx} + \kappa_{yy}) \\ \sigma_{xy} = \frac{E}{1+\nu} (\varepsilon_{xy} + \kappa_{xy}) \end{cases}$$

We thus have the following inverse relation between the strain tensor ε and the stress tensor σ (considering Hooke's law for the thin plate) in matrix form (Eq. 8.14):

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}$$

6. Equilibrium Equations of a Plate

Consider an element $dx \, dy$ of the uniformly loaded plate shown in following figure.

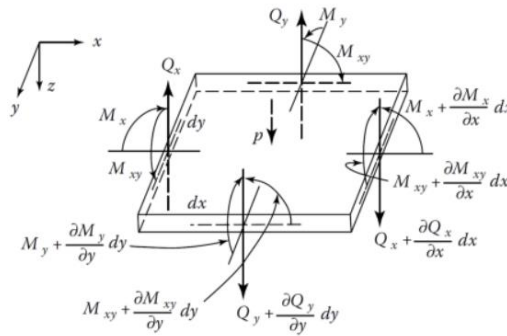


Figure 8.1 – An element of the uniformly loaded plate

The distributed stresses through the plate thickness produce bending moments, torsion moments, and vertical shear forces. The resultants (moments and forces) per unit length can be written as (Eq. 8.15):

$$\begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} z \, dz; \quad \begin{pmatrix} Q_x \\ Q_y \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \tau_{xz} \\ \tau_{yz} \end{pmatrix} dz$$

The differential equation of equilibrium for thin plates takes the form (Eq. 8.16):

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p$$

The differential equation governing the bending of plates can be obtained as (Eq. 8.17):

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}$$

Where:

- w is the deflection.
- The parameter D is the flexural rigidity of the plate: $D = \frac{E h^3}{12(1-\nu^2)}$

This equation, presented by Lagrange in 1811, can be written in compact form as (Eq. 8.18):

$$\nabla^4 w = \frac{p}{D}$$

The Lagrange equation is the governing differential equation of thin plate bending.

The strain energy is defined as the sum of the membrane strain energy and the bending strain energy. The strain energy of the thin plate is given by (Eq. 8.19):

$$W = \frac{1}{2} \int_{A_t} \left[\varepsilon_m^T \frac{E h}{1 - \nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 - \nu \end{pmatrix} \varepsilon_m + \varepsilon_b^T \frac{E h^3}{12(1 - \nu^2)} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 - \nu \end{pmatrix} \varepsilon_b \right] dx dy$$

8. Conclusion

The study of thin plate theory establishes a rigorous mechanical framework by decomposing the global strain tensor into membrane, bending, and shear contributions, enabling a clear description of both in-plane and out-of-plane behavior. The linearized Green–Lagrange strain tensor, combined with Kirchhoff–Love kinematic assumptions, reduces the three-dimensional problem to a two-dimensional one governed by mid-surface displacements and rotations. Through Hooke's law, the constitutive relations link the stress tensor to these deformation components, yielding explicit expressions for bending, torsion, and membrane stresses across the plate thickness. The integration of stresses over the thickness introduces the resultant moments and shear forces, whose equilibrium leads to the governing biharmonic equation — the Lagrange equation — relating the deflection field to the applied transverse load. The flexural rigidity parameter D , dependent on the elastic modulus, Poisson's ratio, and plate thickness, emerges as the key mechanical property controlling the plate's resistance to bending. Finally, the strain energy formulation, unifying membrane and bending contributions in a single variational expression, provides the foundation for numerical methods such as the finite element analysis of thin plate structures.

Exercises

Exercise 1 Simply Supported Rectangular Plate Under Uniform Load

Consider a rectangular thin plate of dimensions $a \times b$ and thickness h , simply supported on all four edges and subjected to a uniform transverse load p_0 . The material has elastic modulus E and Poisson's ratio ν .

1. Write the Lagrange governing equation for this plate.
2. Propose a solution for the deflection $w(x, y)$ in the form of a double Fourier sine series.
3. Determine the maximum deflection w_{max} at the center of the plate.
4. Express the flexural rigidity D .

Exercise 2 Membrane and Bending Stress Distribution Through the Thickness

A thin plate of thickness $h=10$ mm, elastic modulus $E=210$ GPa, and Poisson's ratio $\nu=0.3$ is subjected simultaneously to:

- A membrane strain state: $e_{xx} = 5 \times 10^{-4}$, $e_{yy} = 2 \times 10^{-4}$, $e_{xy} = 0$
 - A bending curvature state: $\kappa_{yy} = 0.01 \text{ m}^{-1}$, $\kappa_{xx} = 0.01 \text{ m}^{-1}$, $\kappa_{xy} = 0$
1. Write the total strain components ε_{xx}^g and ε_{yy}^g as functions of z .
 2. Compute the stresses σ_{xx} and σ_{yy} at the top surface $z = +h/2$ and the bottom surface $z = -h/2$
 3. Identify which contribution (membrane or bending) dominates at the outer fibers.

Exercise 3 Computation of Strain Energy of a Thin Plate Strip

Consider a thin plate strip of width b , length a , and thickness h , clamped at $x=0$ and free at $x=a$, subjected to a uniform bending moment M_0 applied at the free edge. Assume plane strain conditions in the y -direction ($\kappa_{yy} = 0, e_{yy} = 0, e_{xy} = 0$).

Determine the curvature κ_{xx} from the moment-curvature relation.

Write the bending strain energy W_b using equation (8.19).

Compute W_b numerically for: $a=0.5$ m, $b=0.2$ m, $h=8$ mm, $E=200$ GPa, $\nu=0.3$, $M_0 = 500 \text{ N} \cdot \text{m/m}$.

Solutions

Solution to exercise 1

The governing Lagrange equation is:

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_0}{D}$$

For simply supported boundary conditions, the deflection satisfying $w=0$ and $M=0$ on all edges is proposed as:

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

The uniform load is similarly expanded:

$$p_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where the Fourier coefficients for a uniform load are:

$$p_{mn} = \frac{16 p_0}{mn\pi^2} \quad \text{for } m, n \text{ are odd, } p_{mn} = 0 \text{ otherwise}$$

3. Substituting into the governing equation gives:

$$W_{mn} = \frac{p_{mn}}{D \pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}$$

The maximum deflection at the center ($x=a/2, y=b/2$) is dominated by the $m=n=1$ term:

$$w_{max} = \frac{16 p_0}{\pi^6 D} \cdot \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2}$$

4. The flexural rigidity:

$$D = \frac{E h^3}{12(1 - \nu^2)}$$

Solution to exercise 2

1. the total strains are:

$$\begin{aligned} \varepsilon_{xx}^g(z) &= e_{xx} + z \kappa_{xx} = 5 \times 10^{-4} + 0.02 z \\ \varepsilon_{yy}^g(z) &= e_{yy} + z \kappa_{yy} = 2 \times 10^{-4} + 0.01 z \end{aligned}$$

2. Using the constitutive law :

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1 - \nu^2} (\varepsilon_{xx}^g + \nu \varepsilon_{yy}^g) \\ \sigma_{yy} &= \frac{E}{1 - \nu^2} (\nu \varepsilon_{xx}^g + \varepsilon_{yy}^g) \end{aligned}$$

With $\frac{E}{1 - \nu^2} = \frac{210 \times 10^9}{1 - 0.09} = 230.77$ Gpa

- At $z = +h/2 = +0.005$:

$$\varepsilon_{xx}^g = 5 \times 10^{-4} + 0.02 \times 0.005 = 6 \times 10^{-4}$$

$$\varepsilon_{yy}^g = 2 \times 10^{-4} + 0.01 \times 0.005 = 2.5 \times 10^{-4}$$

$$\sigma_{xx}^{top} = 230.77 \times 10^9 (6 \times 10^{-4} + 0.3 \times 2.5 \times 10^{-4}) = 155.8 \text{ Mpa}$$

$$\sigma_{yy}^{top} = 230.77 \times 10^9 (0.3 \times 6 \times 10^{-4} + 2.5 \times 10^{-4}) = 99.6 \text{ MPa}$$

- At $z = -h/2 = -0.005$:

$$\varepsilon_{xx}^g = 5 \times 10^{-4} - 10^{-4} = 4 \times 10^{-4}$$

$$\varepsilon_{yy}^g = 2 \times 10^{-4} - 0.5 \times 10^{-4} = 1.5 \times 10^{-4}$$

$$\sigma_{xx}^{bot} = 230.77 \times 10^9 (4 \times 10^{-4} + 0.3 \times 1.5 \times 10^{-4}) = 106.5 \text{ Mpa}$$

$$\sigma_{yy}^{bot} = 230.77 \times 10^9 (0.3 \times 4 \times 10^{-4} + 1.5 \times 10^{-4}) = 62.3 \text{ MPa}$$

3. The bending contribution $z \kappa_{xx}$ varies linearly through the thickness and reverses sign. The difference in stress between top and bottom ($\Delta \sigma_{xx} \approx 49.3$) reflects the bending contribution, while the average (≈ 131) reflects the membrane contribution. In this case, **membrane stresses dominate** since e_{xx} is large relative to $h \kappa_{xx}/2$

Solution to exercise 1

1. Under the given conditions, the bending moment resultant is:

$$M_x = D \kappa_{xx}$$

So the curvature is:

$$\kappa_{xx} = \frac{M_0}{D}$$

With $D = \frac{Eh^3}{12(1-\nu^2)}$

2. With $\kappa_{yy} = \kappa_{xy} = 0$, the strain energy reduces to:

$$W_b = \frac{1}{2} \int_0^a \int_0^b \varepsilon_b^T \cdot \frac{Eh^3}{12(1-\nu^2)} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \varepsilon_b \, dy \, dx$$

Since only $\kappa_{xx} \neq 0$:

$$W_b = \frac{1}{2} \int_0^a \int_0^b D \kappa_{xx}^2 \, dy \, dx = \frac{1}{2} D \kappa_{xx}^2 a b$$

Substituting $\kappa_{xx} = M_0/D$

$$W_b = \frac{M_0^2 a b}{2 D}$$

3. Numerical evaluation:

$$D = \frac{200 \times 10^9 \times (0.008)^3}{12(1-0.09)} = \frac{200 \times 10^9 \times 5.12 \times 10^{-7}}{10.92} = 9380 \text{ N/m}$$

$$W_b = \frac{(500)^2 \times 0.5 \times 0.2}{2 \times 9380} = \frac{25000 \times 0.1}{18760} = 0.133 \text{ J}$$

This represents the total elastic strain energy stored in the plate strip under uniform bending, recoverable upon unloading.

General Conclusion

This course has provided a comprehensive and rigorous treatment of the theory of elasticity, progressing systematically from the mathematical foundations of continuum mechanics to the analysis of specific structural configurations. The journey from index notation and tensor algebra through the governing equations of linear elasticity, constitutive modeling, and structural applications reflects the unified character of the discipline and its indispensable role in modern structural engineering.

Summary of Core Contributions

The first part of the course established the essential mathematical and physical concepts upon which all subsequent developments rest. The introduction of vector and tensor analysis, index notation, and the Einstein summation convention provided the formal language needed to express field equations in compact, coordinate-independent form. The classification of elastic materials according to their symmetry properties introduced the conceptual spectrum from isotropic to fully anisotropic behavior, establishing the framework for constitutive modeling. The theory of the stress state developed in Chapter Two revealed how internal forces are distributed within a continuous medium and introduced the Cauchy stress tensor as the fundamental mathematical object for describing this distribution. Key results including Cauchy's theorem, the differential equations of equilibrium, principal stresses, stress invariants, and Mohr's graphical representation collectively constitute the core of stress analysis. These tools are applied daily in engineering practice for failure assessment, design verification, and structural optimization.

The complementary theory of the strain state examined in Chapter Three established the kinematic foundations of deformation analysis. The Green–Lagrange strain tensor in finite deformation, its linearization under the small perturbation hypothesis, and the resulting infinitesimal strain tensor together describe how material elements elongate, contract, and distort under loading. The compatibility equations, ensuring the existence of a continuous displacement field, and the presentation of modern experimental measurement techniques including digital image correlation and fiber optic sensors, demonstrated the essential connection between theoretical analysis and experimental validation.

Chapter Four bridged the gap between kinematics and statics through the constitutive laws of elasticity. The generalized Hooke's law, expressed in terms of Lamé constants and engineering elastic moduli, provides the relationship between stress and strain for linear isotropic materials. Extensions to orthotropic and anisotropic materials, the thermoelastic constitutive law, and the concept of strain energy enriched this framework and demonstrated its applicability across a broad range of engineering materials and loading conditions.

The general equations of linear elasticity assembled in Chapter Five—the Navier–Cauchy displacement equations and the Beltrami–Michell stress equations—represent the complete mathematical formulation of the elasticity problem. These partial differential equations, combined with appropriate boundary conditions, govern the mechanical response of any elastic solid. Saint-Venant's principle provided an important engineering insight allowing complex local loading conditions to be replaced by simpler equivalent systems at distances sufficiently far from the point of application.

Structural Applications

The second part of the course demonstrated how the theoretical framework of linear elasticity is applied to practical structural problems. Plane elasticity theory, through the formulations of plane stress and plane strain, reduces the three-dimensional governing equations to two-dimensional problems that can be addressed analytically using the Airy stress function. The biharmonic governing equation resulting from this approach has enabled the derivation of classical closed-form solutions for beams, plates, and structural components with simple geometries.

Beam bending theory, presented in Chapter Seven through both the Euler–Bernoulli and Timoshenko models, illustrates the power of structural idealization in simplifying complex three-dimensional problems. The Euler–Bernoulli model, with its assumption of plane sections remaining plane and perpendicular to the neutral axis, provides accurate and computationally efficient results for slender beams. The Timoshenko model extends this capability to thick beams where shear deformation is significant, making it essential for modern finite element analysis. The comparison between these two theories underscored the importance of selecting appropriate assumptions based on structural geometry and loading conditions.

The study of thin plates in Chapter Eight extended the principles of bending analysis to two-dimensional structural elements. The Kirchhoff–Love plate theory, with its assumptions of small deflections and negligible transverse shear, yields the governing biharmonic Lagrange equation, whose solutions describe the deflection field of plates under transverse loading. The decomposition of plate deformation into membrane and bending contributions, and the formulation of the flexural rigidity parameter, provide the basis for both analytical and numerical approaches to plate analysis.

In conclusion, the theory of elasticity remains as relevant today as when it was first formulated by the great mathematicians and engineers of the nineteenth century. Its integration of mathematical rigor, physical insight, and engineering applicability makes it an enduring cornerstone of structural analysis. Students who have mastered its principles are equipped to approach complex engineering problems with confidence, to use advanced computational tools critically, and to contribute to the continued advancement of the discipline.

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