الجمهورية الجزائرية الديمقراطية الشعبية
People's Democratic Republic of Algeria
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research

جامعة زيان عاشور – الجلفة
Ziane Achour University of Djelfa

كلية العلوم و التكنولوجيا
Faculty of Science and Technology

Department : Electrical Engineering

Order N° : 05 / 2022

Defense authorization N° : …11/2023

DOCTORAL THESIS

3rd Cycle doctoral (D-LMD)

Presented by

Abdesselam TAMRI

With a view to obtaining the doctoral diploma in 3rd Cycle Doctoral (D-LMD)

Branch: Electronics

Specialty: Signal Image and Systems

Topic

Reduction of second-order continuous systems based on Krylov space

Supported, on 03/04 / 2023, before the jury composed of:

Last and first name	Grade	Institution of affiliation	Designation
Mr Omar Mansour	Professor	University of Djelfa	President
Mrs Lahcène MITICHE	Professor	University of Djelfa	Supervisor
Mm Amel Baha Houda ADAMOU-MITICHE	Professor	University of Djelfa	Examiner
Mr Noureddine BOUGUECHAL	Professor	University of Batna 2	Examiner
Mr Abderazak GACEMI	MCA	University of Médéa	Examiner

Acknowledgements

I would first like to thank ALLAH almighty for giving me the strength and then the patience to complete my thesis.

I would like to sincerely thank my supervisor. Pr. Lahcène MITICHE who helped me to do everything from the beginning of the project idea to the last step in it, which enable me to improve and understand the topic. Without his invaluable advices, this work would not have been possible.

I am very grateful to the Pr.ADAMOU-MITICHE Amel Baha Houda, for her encouragement, supervision and support.

I want like to thank the jury members who honored me with their participation in the defense.

I would like to acknowledge and express my deepest gratitude to my supervisors, professor A. B. H Adamou-Mitiche and professor L.Mitiche, who were abundantly helpful and offered invaluable assistance, support and guidance, without which this research project would not have been possible. I want to take this opportunity to thank the people and institutions that assisted in fulfilling this thesis. This thesis would not have been possible without the wholehearted cooperation of the many friends and colleagues who graciously consented to be ready to give me a hand during the work. I wish to thank science and technology Department of Djelfa University and all the administrators, teachers, and special thanks to the members of examining board.

This thesis is dedicated to :

- My parents, brothers and sisters
- My wife for her support and my kids: Abdelkader , Marwa, Doaa , and Yacine

And to anyone who likes me. I hope that Allah will preserve them and give them health and wellness.

All the teachers of the Electrical Engineering Department of the University Ziane Achour of Djelfa and to my colleagues from ATM Mobilis ,

To my friends.

Abdesselam TAMRI

الملخص

يعتبر نموذج ثابت الوقت الخطي (LTI) واسع النطاق من الدرجة الثانية تمثيلً معروفًا لنمذجة السلوك الديناميكي للأنظمة المعقدة متعددة المتغيرات في مختلف مجالات العلوم والهندسة ، مثل الكهربائية والميكانيكية والهيكلية والكهرومغناطيسية والميكرو - .الأنظمة الكهروميكانيكية (MEMS) .تواجه بعض هذه الأنظمة مشاكل حسابية في المحاكاة بسبب ترتيب نموذجها الضخم ، لمعالجة هذه المشكلة .فإنه من المناسب العثور على نموذج تقريبي موثوق به بترتيب مخفض ، والذي يحل محل النموذج الأصلي في المحاكاة أو عنصر التحكم ، والذي يمكن أن يحافظ على بنية الترتيب الثاني للنظام الأصلي ونفس الخصائص الرئيسية للنموذج ذو الترتيب الكامل مثل الاستقرار .في هذه الأطروحة ، تم تقديم خوارزمية جديدة لتقليل الأنظمة الديناميكية ذات الأبعاد الكبيرة من الدرجة الثانية من خلال إجراء تخفيض Arnoldi من الدرجة الثانية (SOAR) مع معيار إيقاف يعتمد على معامل جديد ، يُعرف باسم معامل أداء التصنيف الرقمي (NRPC) ، من أجل الإنهاء المبكر الفعال واختيار الترتيب الأمثل التلقائي للنموذج المصغر. الفكرة الرئيسية لهذه التقنية هي حساب معامل NRPC لكل تكرار لخوارزمية SOAR ، وقياس معلومات التطور الديناميكي للنظام الأصلي ، والتي تضاف إلى كل متجه من فضاء Krylov الفرعي الذي تم إنشاؤه بواسطة خوارزمية SOAR. عندما يتم التحقق من حالة التسامح الديناميكي ، يتم إيقاف الإجراء التكراري للخوارزمية. تم استخدام الأمثلة العددية المطبقة على ثلاث نماذج معيارية للتحقق من فعالية وبساطة الخوارزمية المقترحة.

الكلمات المفتاحية

أنظمة الدرجة الثانية ، فضاء الحالة من الدرجة الثانية ، فضاء فرعي KRYLOV ، الإسقاطات ، نموذج الترتيب المنخفض ، إجراء Arnoldi من الدرجة الثاني (SOAR) ، الحفاظ على الهيكل ، الاستقرار

Résumé

Le modèle linéaire invariant dans le temps (LTI) de second ordre à grande échelle est considéré comme une représentation bien connue pour modéliser le comportement dynamique de systèmes complexes multi variables dans divers domaines de la science et de l'ingénierie, tels que l'électricité, la mécanique, la structure, l'électromagnétisme et la micro -systèmes électromécaniques (MEMS). Certains de ces systèmes rencontrent des problèmes de calcul dans la simulation en raison de leur ordre de modèle énorme, pour traiter ce problème. Il convient donc de trouver un modèle approximé fiable d'ordre réduit, qui remplace le modèle d'origine dans la simulation ou le contrôle, qui peut conserver la structure de second ordre du système d'origine et les mêmes propriétés clés du modèle d'ordre complet telles que la stabilité. Dans cette thèse, un nouvel algorithme est introduit pour réduire les systèmes dynamiques de second ordre de grande dimension grâce à la procédure de réduction d'Arnoldi du second ordre (SOAR) avec un critère d'arrêt basé sur un nouveau coefficient, connu sous le nom de coefficient de performance de rang numérique (NRPC), pour résiliation anticipée efficace et sélection automatique optimale de la commande du modèle réduit. L'idée clé de cette technique est de calculer le coefficient NRPC pour chaque itération de l'algorithme SOAR, et de mesurer les informations d'évolution dynamique du système d'origine, qui sont ajoutées à chaque vecteur du sous-espace de Krylov généré par l'algorithme SOAR. Lorsque la condition de tolérance dynamique est vérifiée, la procédure itérative de l'algorithme est arrêtée. Des exemples numériques appliqués sur trois modèles de référence sont utilisés pour vérifier l'efficacité et la simplicité de l'algorithme proposé.

Mots Clés

Systèmes de second ordre, Espace d’état d’ordre 2, Espaces de Krylov, Projections, Modèle d’ordre réduit , Procédure d'Arnoldi du second ordre (SOAR), Préservation de la structure, Stabilité

Abstract

the large-scale second-order linear time-invariant (LTI) model is considered a well-known representation for modeling the dynamic behavior of multi-variable complex systems in various fields of science and engineering, such as electricity , mechanics, structure, electromagnetism and micro-electromechanical systems (MEMS) some of these systems have computational problems in simulation due to their huge model order. To deal with this problem, it is therefore necessary to find a reliable approximated reduced-order model, which replaces the original model in simulation or control, which can retain the second-order structure of the original system and the same key properties of the original model such as stability. In this thesis a new algorithm is introduced for reducing the large dimensional second-order dynamic systems through the Second Order Arnoldi Reduction (SOAR) procedure with a stopping criterion based on a new coefficient, known as the Numerical Rank Performance Coefficient (NRPC), for efficient early termination and automatic optimal order selection of the reduced model. The key idea of this technique is to calculate the NRPC coefficient for each iteration of the SOAR algorithm, and to measure the dynamic evolution information of the original system, which is added to each vector of the Krylov subspace generated by SOAR algorithm. When the dynamical tolerance condition is verified the iterative procedure of the algorithm is stopped. Numerical examples applied on three benchmark models are used to check the effectiveness and simplicity of the proposed algorithm.

Keywords

Second-order systems, 2nd-order state space, Krylov spaces, Projections, Reduced-order model, Second-order Arnoldi procedure (SOAR), Preservation of structure, Stability

CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ALGORITHMS

LIST OF ACRONYMS

DAE	Differential-algebraic equation
LTI	Linear Time Invariant
MEMS	Micro-Electro Mechanical Systems
MIMO	Multi Input Multi Output
MOR	Model Order Reduction
NRPC	Numerical Rank Performance Coefficient
ODE	Ordinary Differential equation
ROM	Reduced Order Model
SISO	Single Input Single Output
SOAR	Second Order ARnoldi
SVD	Singular Value Decomposition
TF	Transfer Function

LIST OF SYMBOLS

$\mathbb{R}$, $\mathbb{C}$	Fields of real and complex numbers
$\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$	positive and non-negative real numbers
$\mathbb{R}^n$, $\mathbb{C}^n$	vector spaces of real or complex valued tuples of length n
$\mathbb{R}^{nXm}$, $\mathbb{C}^{nXm}$	vector spaces of real or complex valued matrices with n rows and m columns
i	Imaginary unit ($i^2 = -1$)
Re(z), Im(z)	real and imaginary parts of the complex number $z = Re(z) + iIm(z)$
I_n	identity matrix of size n
A^T A^H	transposed of the matrix A
span (A)	linear subspace spanned by the columns of the matrix A
A^{-1}	inverse of the regular matrix A
A^{-T}, A^{-H}	inverse of A^T or A^H, respectively
$\Lambda(A)$	spectrum of the matrix A
$\sigma_{max}(A)$	largest singular value of the matrix A
$\|.\|_2$, $\|.\|_\infty$	$\mathcal{H}_2$, $\mathcal{H}_\infty$ norms
G_L	Transfer function

General Introduction

General Introduction

1. Motivation

The mathematical modeling of most physical systems, such as telecommunication systems, transmission lines and chemical reactors, results in infinite dimensional models. Using engineering tools, we can still roughly represent those systems with approximate finite dimension models [1]. However, complex large-scale systems usually require dimensional models to well represent them. Analysis, simulation and design methods based on this high order model may eventually lead to complicated control strategies requiring complex logic or large amount of computation. Model Order Reduction (MOR) is a branch of systems and control theory which studies properties of dynamical systems in order to reduce their complexity while preserving their input - output behavior [2].

A reduced description and simulation of the dynamics of a physical problem is known as model order reduction. Precision and computational cost are strongly connected. As a result, we must consider this connection as well as the needed precision when selecting a model order reduction strategy. A reduction of models makes it possible to reduce the computation time and reduce the amount of memory used, and this goes through the reduction of the number of necessary state variables.

In the literature, there are a number of model reduction strategies with trade-offs between various relevant factors including computing cost, accuracy, and stability. Balanced Truncation [3] and Hankel Norm Approximation [4] are two of these strategies that ensure stable approximations and attain a priori error bounds. They do, however, necessitate the solution of Lyapunov equations, which renders them inefficient for large-scale systems. For model reduction, many iterative algorithms based on Krylov subspace projection [5] have been widely employed. These methods are computationally efficient because they only use matrix-vector multiplications and take use of sparsity, which is prevalent in large-scale systems. However, several aspects of the original system, such as stability, passivity, and minimum phase character, might be lost when using Krylov-based approximations.

The large-scale 2nd order model is considered a well-known representation for modeling the dynamical behavior of multivariable complex systems in various fields of science and engineering, such as electrical, mechanical, structural, electromagnetic, and micro-electromechanical systems (MEMS). Some of these systems encounter computational problems in simulation due to their huge model order, to treat this problem. Therefore, we desire to find a reliable approximated model with reduced order, which replace the original model in the simulation or the control, which can preserve the second order structure of the original system and the same key properties of the full order model such as stability [6,7].

2. State of the art

The problem of model order reduction for second-order linear systems is basically as old as the topic of the model order reduction itself. This amounts to the relevance of mechanical systems in practical application. Modal truncation, as one of the oldest model reduction methods [12], was quickly extended to the second-order settings in various ways [13,14,15]. Even nowadays, structure-preserving modal truncation is the preferred approach for model reduction in engineering sciences due to its generality and computational simplicity [16]. However, a general problem of related approaches is the selection of the appropriate system modes to approximate the original dynamics. The dominant pole algorithms [17] were developed as remedy to this problem, which in recent years were extended to large-scale spare systems [18,19] as well as to the general case of second-order systems; for example see in [20,21]. In practice, the modeling of internal damping of mechanical systems is often simplified to the use of combination of the mass and stiffness terms of the system leading to so-called modally damped mechanical systems. This subclass of linear mechanical systems holds several advantageous properties that are currently not considered in theory or implementations of structure preserving dominant pole algorithm. This point will not be discussed in this thesis, while also treating other modal truncation concerning bounds for the approximation error and the limited approximation quality.

In this thesis, a model order reduction of the 2nd order system using a Krylov-base technique called Second Order Arnoldi (SOAR) algorithm was discussed. The SOAR approach has attracted many researchers in the last few years, which has been used to solve the following problems: a quadratic Eigen-value [8] [9] , the MOR of second-order dynamical systems [6] [10] [11] and in the analysis of structural acoustics. From mathematical point of view, the SOAR design is based on a projection-based MOR technique that uses a second-order Krylov subspace and the SOAR procedure to generate the projection matrix as follows: in the first step

a recurrence formula is defined for the two matrices coefficient A and B and one or two initial vectors, then in the second step we generate an orthonormal basis of projection sub-spaces from the famous second-order Krylov subspace defined in the recurrence formula. Obviously, the SOAR technique is used in MOR, which construct another reduced second-order state-space system with reduced order, where the input-output behavior dynamics are completely recovered i.e. preserving the basic characteristics of the full order system [6][7].

3. Outline of thesis

In this study, we propose an new automated technique to generate the best reduced order model for a large second order system using the SOAR procedure, by defining a new criterion to auto-stop the iteration process in SOAR procedure and to auto-select an acceptable reduced order of the projection matrix, which in the limit of our knowledge is a new proposed idea, the efficiency and robustness of the proposed algorithm is validated by various well-chosen numerical examples of 2^{nd} order models.

This thesis is organized as follow:

- The first chapter is a background on Linear Time invariant (LTI) systems and Model Order reduction (MOR).
- The second chapter presents model order reduction based on Krylov sub-space technique.
- The third chapter presents the Second Order Arnoldi method and the theoretical framework used as the basis of the stopping criterion for auto selection of the reduced model.
- Chapter four shows the validation of the proposed technique in various performed tests.
- The fifth chapter is the conclusion drawn in this work.

Chapter One

LTI System Theory and Fundamentals

Chapter I : LTI System Theory and Fundamentals

I.1 Introduction

In this chapter, we introduce the basic concepts of linear time-invariant systems. The content of this chapter is mainly taken from [8], it can also be found in other standard textbooks about systems theory or model order reduction; see, e.g. [9–13]. This section itself is additionally separated into the classical first-order systems and second-order systems. This chapter is concluded in section 1.4 by two examples of second-order from mechanical engineering.

I.2 First-order systems

Before the special case of the second-order systems is considered, some properties of the *first-order linear time-invariant (LTI) systems* are needed first. These systems have the form

$$\begin{cases} E\dot{x}(t) = A\,x(t) + B\,u(t), \\ y(t) = C\,x(t) + D\,u(t), \end{cases} \quad \text{(I.1)}$$

This form is called a generalized state space of a linear time-invariant system.

With $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{N,N}$, $\mathbf{B} \in \mathbb{R}^{N,m}$, $\mathbf{C} \in \mathbb{R}^{p,N}$, and $\mathbf{D} \in \mathbb{R}^{p,m}$. $x(t)$ is called the state vector; its dimension $N \in \mathbb{N}$ denotes the *order* of the model. $u(t)$ and $y(t)$ contain the input and output signals of the system, respectively. Systems with $m,p \geq 2$ are referred to as multi-input multi-output (MIMO) systems; the special case $m=p=1$ is called single-input single-output (SISO) system. Default assumptions for (I.1) in model order reduction are $x(t_0) = 0$ and $t_0=0$ (initial condition) to neglect the initial value's influence on the system's behavior. These assumptions are also made through-out this thesis. The behavior of (I.1) is given via the three-dependent functions: $u: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$, the inputs that are used to control $x: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^N$, the internal states, to get the desired outputs $y: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^p$.

When the term $\mathbf{D} \neq \mathbf{0}$ we call it a feed-through term. This feed-through term will not play any role in this thesis, but all developed model reduction theory can be transferred to systems with feed-through term by preserving the original term in the reduced-order system $\widehat{D} = D$.

In some applications, the case $\widehat{D} \neq D$ is of particular interest. This can be treated in certain model reduction approaches, like interpolation methods, by additional modifications of the construction formulae; see, e.g. [14,15]

In the literature, the first-order system (I.1) can be found under different names, often, depending on the specific realization of the **E** matrix. The system (I.1) is called a *standard state-space system* in case of **E= I_N** and it is called *a generalized state-space system* if **E** is invertible but not the identity, i.e., when the states are described by a system of Ordinary differential equations (ODEs) with a mass matrix. In case of **E** singular, (I.1) contains Differential-algebraic equations (DAEs) and is referred to as *descriptor system*.

Definition1.1 (System realization and order)

The quadruple $\sum = (A, B, C, E)$ *and* $\sum = \left[\begin{array}{c|c} E,A & B \\ \hline C & 0 \end{array}\right] \in \mathbb{R}^{N.N} \times \mathbb{R}^{N.m} \times \mathbb{R}^{p.N} \times \mathbb{R}^{N.N}$ is called a *realization* of the system (I.1). The *order* of (I.1) is defined to be the dimension of the corresponding state-space N.

In general, the realization of a system is not unique in the sense of the input-to-output behavior, i.e., the same system can be described by different realizations. A system realization (I.1) is called *equivalent* to another realization $\widetilde{\sum} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{E})$ if and only if there exist (invertible) transformation matrices Z, T $\in \mathbb{C}^{N.N}$ such that

$$\widetilde{\boldsymbol{E}} = \boldsymbol{Z^H E T},\ \widetilde{\boldsymbol{A}} = \boldsymbol{Z^H A T},\ \widetilde{\boldsymbol{B}} = \boldsymbol{Z^H B},\ \widetilde{\boldsymbol{C}} = \boldsymbol{C T} \quad \text{(I.2)}$$

Therein, the matrix **T** yields a coordinate transformation $\tilde{x} = Tx$ and **Z** transforms the describing equations. The change of one system to an equivalent one in the sense of (I.2) is referred to as *generalized state-space transformation.*

The following definition introduces some important system properties.

Definition 1.2 (Basic system properties)

The system (I.1) is called :

(a) *Asymptotically stable* or *c-stable*, if all eigenvalues of the matrix pencil λE−A, i.e., all $\lambda \in \mathbb{C}$ such that det(λE−A) = 0, have negative real parts

(b) *Controllable* in $[t_0,t_f]$, if any initial state $x(t_0)$ can be steered to any final state $x(t_f)$ by an appropriate input signal $u(t)$ with finite energy.

(c) *Observable* in $[t_0,t_f]$, if the set of states such that $y(t) = Cx(t)=0$, for all $t \in [t_0,t_f]$, contains only the zero state $x(t)=0$.

Controllability and observability are important concepts in model order reduction to characterize system components that do not contribute substantially to the input-to-output behavior of the system. It can be shown that a system (I.1) is *minimal*, i.e., has the smallest possible order to describe exactly the input-to-output behavior, if and only if it is controllable and observable. There are variety of different equivalent definitions and criteria for the system properties in Definition 1.2. Some can be found, for example, in [8].

A useful tool to deal with systems of differential equations is the Laplace transform. For a time domain function $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$, its *Laplace transform* is defined to be

$$F(s) = L\{f(t)\}(s) := \int_0^\infty f(t)\, e^{-st}\, dt, \quad \text{(I.3)}$$

If the integral exists, with the complex frequency variable $s \in \mathbb{C}$. Applying (I.3) to the linear system (I.1) results in an equivalent description in the complex frequency domain via algebraic equations rather than differential ones

$$\begin{aligned} sEx(s) - Ex_0 &= AX(s) + BU(s), \\ Y(s) &= CX(s), \end{aligned} \qquad \text{(I.4)}$$

Where X: $\mathbb{C} \rightarrow \mathbb{C}^n$, U:$\mathbb{C} \rightarrow \mathbb{C}^p$ are the Laplace transform off the equally named time domain function x, u, and y, respectively. With the assumption that $x_0 = 0$, the input-to-output behavior of (I.1) in the frequency domain can be directly described by

$$Y(s) = (C(sE - A)^{-1}B)U(s) = G_L(s)U(s) \quad \text{(I.5)}$$

Where the complex, matrix-valued function

$$G_L = C(sE - A)^{-1}B \quad \text{(I.6)}$$

Is called the *transfer function* (*TF*) of (I.1)

In model order reduction, the input-to-output behavior (I.1) is approximated via a surrogate model of smaller order. For an analysis of the approximation quality, norms for dynamical systems are needed. The following definition states two commonly used system norms.

Definition 1.3 (System norms)

Assume (I.1) to be asymptotically stable with its transfer function (I.6).

(a) The $\mathcal{H}_2$-norm is defined as

$$\|G_L\|_{H_2} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G_L(\omega i)\|_F^2 \, d\omega}$$

(b) The $\mathcal{H}_\infty$-norm is defined as

$$\|G_L\|_{H_\infty} = \sup_{\omega \in \mathbb{R}} \|G_L(\omega i)\|_2$$

While most of the time, the norms in Definition 1.3 are sufficient for studying stable systems, it should be noted that an important expansion of the $\mathcal{H}_\infty$-norm for systems with anti-stable parts, i.e., where eigenvalues of $\lambda E - A$ have positive real parts, is the $\mathcal{L}_\infty$-norm . this norm is analogously to the $\mathcal{H}_\infty$-norm defined as

$$\|G_L\|_{L_\infty} = \sup_{\omega \in \mathbb{R}} \|G_L(\omega i)\|_2 \qquad \text{(I.7)}$$

The norms in Definition 1.3 are defined using the system's transfer function in the frequency domain. Results from Parseval, Plancherel and Payley-Wiener give links between the time and frequency domain description of (I.1) in terms of norms and spaces [8]. Roughly speaking, the approximation behavior of the transfer function in frequency domain is equivalent to the input-to-output approximation behavior in the time domain, i.e., the better the transfer function is approximated the smaller the time domain input-to-output error will be. In fact, the following two inequalities can be shown to hold in time domain (and in frequency domain with according changed functions and spaces)

$$\begin{aligned} \|y - \hat{y}\|_{L_2} &\leq \left\|G_L - \hat{G}_L\right\|_{H_\infty} \|u\|_{L_2}, \\ \|y - \hat{y}\|_{L_\infty} &\leq \left\|G_L - \hat{G}_L\right\|_{H_2} \|u\|_{L_2} \end{aligned} \qquad \text{(I.8)}$$

For u and $y - \hat{y}$ in the appropriate spaces, and where $\hat{y}$ is the output signal of an approximating system corresponding to $\hat{G}_L$. The two norms used above are the time domain *L_2-I and L_∞-* norms, which are defined by

$$\|x\|_{L_2} = \sqrt{\int_{t_0}^{t_f} \|x(t)\|_2^2 \, dt,} \qquad \text{(I.9)}$$

$$\|x\|_{L_\infty} = \sup_{t \in [t_0, t_f]} \|x(t)\|_\infty \qquad \text{(I.10)}$$

For a time domain function x: $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$.

I.3 Second-order systems

The main interest of this thesis lies in mechanical systems. In the LTI case, these systems are usually described by differential equations with second-order time derivatives of the form

$$G_L: \begin{cases} M\ddot{x}(t) + E\dot{x}(t) + K\,x(t) = B_u u(t), \\ y(t) = C_p x(t) + C_v \dot{x}(t) \end{cases} \qquad \text{(I.11)}$$

With $M, E, K \in \mathbb{R}^{NxN}$, $B_u \in \mathbb{R}^{Nxm}$, $C_p, C_v \in \mathbb{R}^{pxN}$; M invertible, if not stated otherwise, and the initial conditions $x(t_0) = x_{p,0}$, $\dot{x}(t_0) = x_{v,0}$ with $x_{p,0}\, x_{v,0} \in \mathbb{R}^N$.

Systems of the form (I.11) are further on referred to as *second-order LTI systems.* The system matrices *M, E, K* are thereby known as *mass, damping, and stiffness matrices* . conform with the previous section, the default assumption for systems like (I.11) will be zero initial conditions $x_{p,0} = x_{v,0} = 0$, with $t_0 = 0$. The definition 2.1 is extended appropriately for (I.11). the order of (I.11) is the corresponding state-space dimension *N*, and the tuple

$$G_L = (M, E, K, Bu, C_p, C_v)$$

is the realization of (I.11). In case of mechanical systems, *M* and *K* are usually symmetric positive definite and $E+E^T$ symmetric positive semi-definite. Often also *E* itself is symmetric positive semi-definite.

In principal, the theory of linear first-order systems (I.1) can be directly transferred to the second-order case by reformulating (I.11) as a first-order system. There exist infinitely many first-order realizations of (I.11). The most commonly used ones are summarized in the following; see, e.g. [16,17].

The first companion form realization can be obtained by introducing the first-order state vector $x^T = [x^T \;\; \dot{x}^T]$. Reordering the lower-order dynamics to the right-hand side yields an equivalent description of (I.11) by a first-order system of the form (I.1), with the system matrices

$$E_{fc} = \begin{bmatrix} J_{fc} & 0 \\ 0 & M \end{bmatrix},\; A_{fc} = \begin{bmatrix} 0 & J_{fc} \\ -K & -E \end{bmatrix},\; B_{fc} = \begin{bmatrix} 0 \\ B_u \end{bmatrix},\quad C_{fc} = [C_p \quad C_v] \qquad \text{(I.12)}$$

Where $J_{fc} \in \mathbb{R}^{NxN}$ is an arbitrary invertible matrix. The input-to-output behavior of (I.11) and the first-order system (I.1) with the matrices (I.12) is identical. A classical choice for the invertible matrix is $J_{fc} = I_N$. In case of *M, E, K* symmetric and *K* invertible, another suitable choice for the invertible matrix is $J_{fc} = -K$, since thereby E_{fc} and A_{fc} become symmetric. If

additionally $B_u = C_v^T$ and $C_p = 0$ hold, the first companion form realization is also state-space symmetric.

A different realization is obtained by moving only the state without time derivative to the right-hand side. The *second companion form realization* of (I.11) is then given by

$$E_{sc} = \begin{bmatrix} E & M \\ J_{sc} & 0 \end{bmatrix},\ A_{sc} = \begin{bmatrix} -K & 0 \\ 0 & J_{sc} \end{bmatrix},\ B_{sc} = \begin{bmatrix} B_u \\ 0 \end{bmatrix},\quad C_{sc} = [C_p \quad C_v] \quad \text{(I.13)}$$

With $J_{sc} \in \mathbb{R}^{NxN}$ an arbitrary invertible matrix. The default choice for J_{sc} in (I.13), if M is invertible, is $J_{sc} = M$. Then, in case of M, E, K symmetric, the first-order system matrices E_{sc} and A_{sc} become symmetric, too. Also, the second companion from realization becomes state-space symmetric if additionally $B_u = C_p^T$ and $C_v = 0$ hold.

Since (I.12) and (I.13) are both realization of the same second-order system, i.e., they are equivalent, the question off the corresponding transformation matrices in (I.2) arises to switch between the two realizations. One can easily prove that (I.12) can be transformed into (I.13) using the transformation matrices

$$Z_{fc2sc} = \begin{bmatrix} J_{fc}^{-T} E^T & J_{fc}^{-T} J_{sc}^T \\ I_N & 0 \end{bmatrix} \quad and \quad T_{fc2sc} = \begin{bmatrix} I_N & 0 \\ 0 & I_n \end{bmatrix} = I_{2N} \quad \text{(I.14)}$$

i.e., it holds

$$E_{sc} = Z_{fc2sc}^T E_{fc} T_{fc2sc},\ A_{sc} = Z_{fc2sc}^T A_{fc} T_{fc2sc},\ B_{sc} = Z_{fc2sc}^T B_{fc},\ C_{sc} = C_{fc} T_{fc2sc} \quad \text{(I.15)}$$

Note that the reverse transformation from second to first companion form is given by the inverse transformation matrices

$$Z_{fc2sc}^{-1} = \begin{bmatrix} 0 & I_N \\ J_{sc}^{-T} j_{fc}^T & -J_{sc}^{-T} E^T \end{bmatrix} \quad \text{(I.16)}$$

In practice, while both companion forms have different advantages, they can quickly run into numerical problems during computations due to the indefiniteness of the first-order system matrices. Therefore, a third first-order realization is mentioned can be also used.

Assuming K to be invertible, the *strictly dissipative realization* of (I.11), as introduced in [16], is given by

$$E_{sd} = \begin{bmatrix} K & \gamma M \\ \gamma M & M \end{bmatrix}, \quad A_{sd} = \begin{bmatrix} -\gamma K & K-\gamma E \\ -K & \gamma M - E \end{bmatrix}, \quad B_{sd} = \begin{bmatrix} \gamma B_u \\ B_U \end{bmatrix}, \quad C_{sd} = [C_p \quad C_v], \tag{I.17}$$

With the parameter$0 < \gamma < \lambda_{min}(E\left(M + \frac{1}{4}EK^{-1}E\right)^{-1}$. It was shown in [16] that in case of mathematical systems with *M*, *E*, *K* symmetric positive definite, this realization is strictly dissipative, i.e., E_{sd} is symmetric positive definite and $A_{sd} + A_{sd}^T$ is symmetric negative definite. Using the realization (I.17) gives numerical advantages in computational methods that work with projected spectra of $\lambda E - A$ rather than directly with the second-order system matrices. But applying (I.17) comes with the cost of increased computational complexity as there are no zero blocks in the matrix structure to make use of in computational operations, in contrast to (I.12) and (I.13).

As before, the strictly dissipative realization (I.17) is equivalent to the other two realizations (I.12) and (I.13) such that again the question of appropriate transformation matrices to switch between the realization need to be answered. While in [16] only the transformation into (I.12) with a specific choice for J_{fc} namely $J_{fc} = K$, was shown, it can be observed that with

$$Z_{f_c 2 s_d} = \begin{bmatrix} J_{fc}^{-T} K^T & \gamma J_{fc}^{-T} M^T \\ \gamma I_N & I_N \end{bmatrix} \; and \; T_{f_c 2 s_d} = I_{2N} \tag{I.18}$$

The more general case holds

$$E_{sd} = Z_{f_s 2 s_d}^T E_{fc} T_{f_c 2 s_d}, \; A_{sd} = Z_{f_c 2 s_d}^T A_{fc} T_{f_c 2 s_d}, \; B_{sd} = Z_{f_c 2 s_d}^T B_{fc}, \; C_{sd} = C_{fc} T_{f_c 2 s_d}. \tag{I.19}$$

The inverse transformation is given by

$$Z_{f_c 2 s_d}^{-1} = \begin{bmatrix} (K - \gamma^2 M)^{-T} J_{fc}^T & -\gamma (K - \gamma^2 M)^{-T} M^T \\ -\gamma (K - \gamma^2 M)^{-T} J_{fc}^T & (K - \gamma^2 M)^{-T} K^T \end{bmatrix} \; and \; T_{f_c 2 s_d}^{-1} = I_{2N} \tag{I.20}$$

With the additional assumption that $K - \gamma^2 M$ is invertible. The transformation of the strictly dissipative realization into the second companion form realization follows then by applying (I.14) or (I.16) to the transformations above.

As in the first-order case, realization of second-order systems are an important point for the application of model reduction methods. In general, the realizations of two second-order systems $G_L \; and \; \tilde{G}_L$ are equivalent if only if there exist *Z*, $T \in \mathbb{C}^{RxR}$, with *R = 2N*,such that corresponding first-order realization of $G_L \; and \; \tilde{G}_L$ are equivalent.

This equivalence is in a certain sense unhandy due to the resulting difficult conditions on the transformation matrices to preserve the second-order structure. A more applicable special case of second-order system equivalence is given in the next definition

Definition 1.4 (Restricted system equivalence [17]**)**

Two second-order systems

$$G_L = \left(M, E, K, B_u, C_p, C_v\right) \; and \; \tilde{G}_L \; = (\tilde{M}, \tilde{E}, \tilde{K}, \tilde{B}_u, \tilde{C}_p, \tilde{C}_v)$$

are called *restricted equivalent*, if there exist transformation matrices $Z, T \in \mathbb{C}^{NxN}$ such that

$$\begin{matrix} M = Z^H \tilde{M} T, \;\; E = Z^H \tilde{E} T, \;\; K = Z^H \tilde{K} T, \\ B_u = Z^H B_u, \;\; C_p = \tilde{C}_p t, \;\; C_v = \tilde{C}_v T \end{matrix} \quad \text{(I.21)}$$

hold. The change between two second-order system realization in the sense of (I.21) is called *restricted state—space transformation.*

It can be shown that the restricted system equivalence is a special cases of the general equivalence of second-order systems by observing that (I.21) is obtained by setting

$$\tilde{Z} = \begin{bmatrix} Z_{11} & 0 \\ 0 & Z \end{bmatrix} \; and \; \tilde{T} \; = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \quad \text{(I.22)}$$

As a generalized state-space transformation (I.2) to first companion form realization (I.12), where $Z_{11} \in \mathbb{C}^{RxR}$ is an arbitrary invertible matrix.

Analogously to first-order case, second-order systems can equivalently described in the frequency domain. Applying the Laplace transform (I.3) to (I.11) yields

$$\begin{matrix} s^2 MX(s) - sMx_{p,0} - Mx_{v,0} = \; -sEX(s) + Ex_{p,0} - KX(s) + B_u U(s), \\ Y(s) = \; C_p X(s) + sC_v X(s) - \; C_v x_{p,0} \end{matrix} \quad \text{(I.23)}$$

Using the assumption $x_{p,0} = Mx_{v,0} \; = 0$ and reordering the terms to get a direct input-to-output relation in the frequency domain results in the second-order transfer function

$$G_L(s) \; = (C_p \; + s \; C_v)(s^2 M + sE + K)^{-1} B_u \quad \text{(I.24)}$$

With the complex variable $s \in \mathbb{C}$. Note that equivalently, inserting any first-order realization (I.11), e.g., (I.12), (I.13) and (I.17), into thee first-order transfer function formulation (I.6) also results in (I.24).

While most system properties of second-order systems are only characterized for their first-order form, e.g., controllability and observability, the concept of asymptotic stability easily transfers to the second-order case: A second-order system (I.11) is asymptotically stable (c-stable) if and only if all eigenvalues λ of the quadratic matrix pencil $\lambda^2 M + \lambda E + K$, i.e., all $\lambda \in \mathbb{C}$ such that $(\lambda^2 M + \lambda E + K) = 0$, have negative real parts.

I.4 Motivating examples for mechanical systems

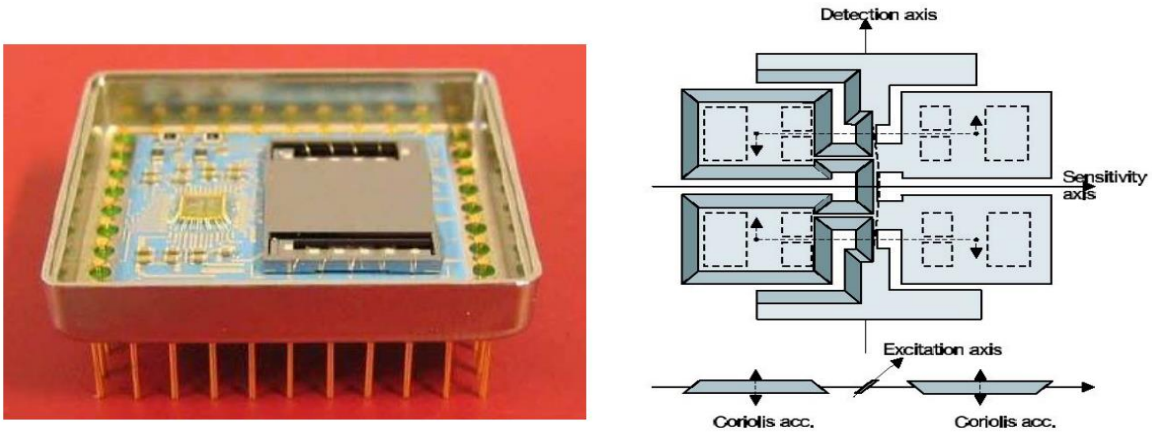
In this section tow motivating examples with underlying mechanical systems are used to illustrate the necessity of structure-preserving model order reduction in practical applications.

I.4.1 Butterfly gyroscope

The butterfly gyroscope is an open benchmark example for model order reduction methods from Oberwolfach Benchmark Collection [18,19]. It models a vibrating micro-mechanical gyroscope for the use in inertial navigation applications. The design of the chip itself is illustrated in Figure 1.1. The displacement field is described by linear three-dimensional partial differential equations from elastodynamics involving second-order time derivatives. Using a spatial finite element discretization yields a linear mechanical system of the form (I.11) described by N = 17361 ordinary differential equations. The states are excited by a single input (m =1) and measuring the displacement of the four wings in the three spatial directions gives $p = 12$ outputs. The internal damping behavior of the gyroscope is modeled Rayleigh (or proportional) damping $E = \alpha M + \beta K$, with the coefficients $\alpha = 0 \; and \; \beta = 10^{-6}$.

In the practical process of improving the butterfly gyroscope, the mechanical system needs to be simulated many times with different input signals to analyze the system's behavior with respect to important physical phenomena, for example, its sensitivity to shocks and vibration. To perform the design process in a reasonable amount of time, it is essential to improve the simulation efficiency of the system. A remedy is the reduction of the number of describing/defining ordinary differential equations by model order reduction techniques. Thereby, the second-order system structure needs to be kept for the analysis process, and is even more beneficial if additional mechanical properties like the symmetry and definiteness of

the system matrices are preserved. Therefore, structure-preserving model order reduction are required here.



(a) Manufactured specimen. (b) Schematic Layouts.

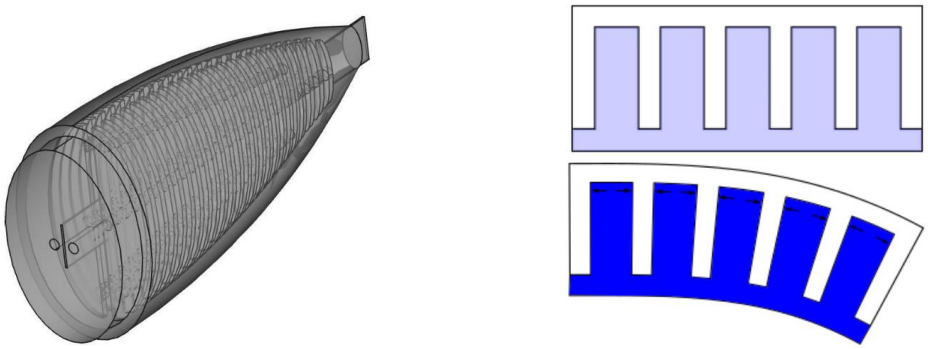
Figure 1.1 Design of the butterfly gyroscope.

I.4.2 Artificial fishtail model

Autonomous underwater vehicles are an important and essential tool in environmental observation tasks [20]. The classical thruster-driven approach has been proven to be mostly inefficient and expensive [21], especially compared with the agile, fast and efficient locomotion that fish naturally developed by evolution [22]. For the construction of fish-like underwater vehicles, the artificial fishtail model was developed [22–24]. Three-dimensional partial differential equations are used to describe the deformation of a fishtail-aped silicon structure; see Figure 1.2a. For the fish-like locomotion, the fluid elastomer actuation principle is used [25]. Therefore, the fishtail consists of two symmetric, ribbed chambers, as shown in Figure 1.2a, which are alternately put under pressure; see Figure 1.2b. this bends the fishtail alternately into the corresponding directions leading to the typical “flapping” behavior that fish use for locomotion.

The fishtail has a complicated geometric structure, which is expressed in discretization of the describing partial differential equations. Using the finite element methods, the discretized equations are given by the linear mechanical system (I.11) with $N = 779232$ ordinary differential equations. A single input $m=1$ is used to describe the pressure flow between the

inner chambers and the displacement of the fish tail's tip is observed in all three spatial directions $p = 3$. The internal damping behavior is modeled via the Rayleigh approach with $E = \alpha M + \beta K$, where $\alpha = 10^{-4}$ and $\beta = 2.10^{-4}$. The size of the resulting system leads to a tremendous amount of computational resources needed to perform simulation, e.g., the simulation of 2s of the fishtails behavior easily takes around 45 min of real-world computation time on computer with (CPU Intel i5 @ 2.10 GHz with 2x8 Cores and RAM 192 GB). The full-order system is simply unbearable when it comes to real-time applications or the use of not so powerful hardware for computation, like an onboard chip. Therefore, structure-preserving model reduction is needed here to provide suitable surrogate model described by only a few differential equations.



(a) Transparent sketch. (b) Fluid chambers in relaxed and pressurized state.

Figure 1.2 Design and actuation principle of the artificial fishtail.

I.5 Conclusion

In this chapter, we presented an overall of the basics of system-theoretic notions and concepts for both first and second order linear systems. We introduced the fundamental concepts and terms associated with Linear Time-Invariant systems such as the state-space representation and some basic system properties, we reviewed some system norms. As this work is focused on the second-order systems, two examples from mechanical engineering were presented in the last section of this chapter. This chapter can serve as starting point for our thesis objective.

Chapter Two
Model Order Reduction

Chapter II: Basics of Model Order Reduction

II.1 Introduction

For a big system, such as the two models examined in the previous chapter, the task of simulation or control design can be expensive in term of time and computing cost. Model order reduction (MOR) is a wide field that is concerned with finding a lower dimensional system, whence faster to simulate, that gives a very good approximation to the original system. Each method finds an approximating subspace that spans the major characteristics of a system's response, a projection matrix is also found that is used to project the larger system onto the smaller subspace. This chapter discusses the theory background concerning model order reduction methods of first and second order systems. First, model order reduction techniques for LTI systems and their classification are reviewed. In section, II.2 the projection framework for model order reduction is established as the main construction approach for reduced-order model in this work. Thereafter, state-of-the-art methods in balanced truncation and Krylov subspaces form first- and second-order systems are presented.

II.2 Model Order Reduction for LTI Systems and their Classification

The numerical simulation of predictive models is a key tool in the design of complicated high-tech systems. These dynamical models, however, are often of high order; a huge number of ordinary differential equations characterizes them. This is due to the system's intrinsic complexity or the discretization of partial differential equations. Model Order Reduction (MOR) is a broad topic concerned with finding a smaller dimensional system, thus quicker to solve, that offers a very good approximation to the original system. Model order reduction techniques have been developed to reduce the complexity of a model while preserving its input-output behavior as much as possible. Other criteria are also used to characterize the reduction method, such as the preservation of essential features of the original model. For example, stability and passivity.

The reduction of the complexity of the models is done by eliminating states from the original model. The elimination of states implies, in practice, a decrease in the dimension of the

matrices, which constitutes the order of the model. Thus, the reduction in order is a dimension reduction and not a reduction in the degree of the derivation of the differential equations defining the model; Figure 2.1 shows the concept in a graphical easy-to-understand way, demonstrating that sometimes very little information is needed to describe a model. This example with pictures off the Stanford bunny shows that, even with only a few facets, the rabbit can still be recognized as such (Graphics credits : Harvard University, Microsoft Research).

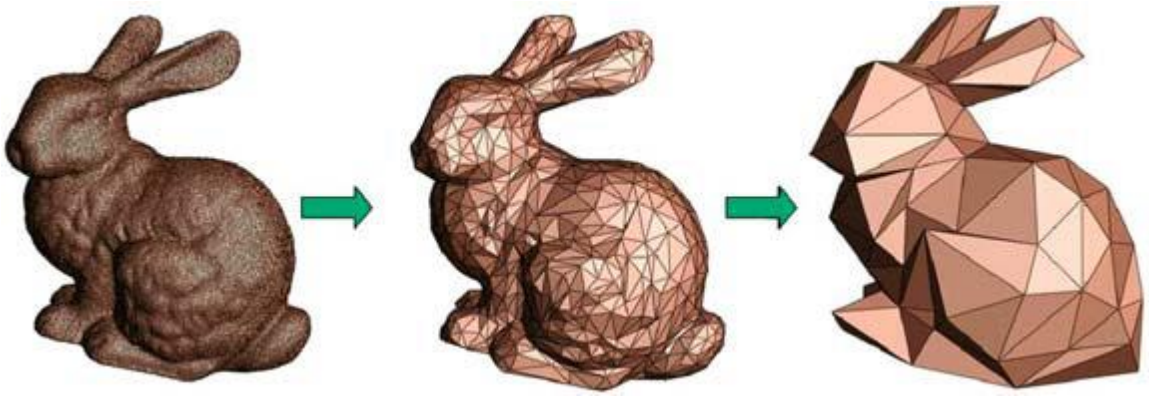


Figure 2. 1 Graphical illustration of model order reduction (source: [26])

There are two widely known model reduction classes for LTI systems, which are now in use, Singular Value Decomposition (SVD) based methods and Krylov Sub-space (moment matching) based methods. The key stages in the first technique are the calculation of the so-called Hankel singular values and balancing of the system. One of the most prominent methodology of the SVD-based techniques is the so-called Balanced truncation model reduction[3,8,26,27]. This approach has good system theoretical qualities including stability preservation and error bound computing. They are not, however, suitable for large-scale systems[28].

The Krylov-based model reduction is an important class of numerical approaches for model reduction. Unlike SVD-based approaches, the reduced model created by Krylov methods is not guaranteed to be stable, and there is no a priori error bound. These techniques, on the other hand, are numerically robust and may be implemented repeatedly, see, for example [5,29–37] for efficient Krylov-based methods implementation. When compared to other reduction classes, the key advantage of this strategy is that it needs less computing effort and memory storage.

The following table shows the different reduction methods for LTI systems and their classification for 1st and 2nd order systems.

Method / System Type	SVD-Based	Krylove-Based
First Order or State-Space Model	Balanced Truncation Hankel Approximation	Realization Interpolation Lanczos Arrnoldi
Second Order Model	Second Order Balanced Truncation	Second Order Arnoldi

Table 2. 1 Model Order Reduction techniques.

II.3 The concept of projection and its application on model order reduction

II.3.1 Petrov-Galerkin Approximation

As we have seen in the preceding section, the modelisation of a system for real-life application can lead to high-dimensional models, i.e., state dimension *n* reaching up to 10^6. In what follows, we give a compact review of the projection concept which is a unifying feature of the most important MOR techniques that can be found in, e.g., [8,26,38–40]. Mathematically speaking, we want to replace system (I.1) by the following one

$$\hat{\Sigma}: \begin{cases} \hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t), \ \hat{x}(0) = \hat{x}_0 \end{cases} \qquad \text{(II.1)}$$

Where $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}, \hat{B} \in \mathbb{R}^{\hat{n}\times m}, \hat{C} \in \mathbb{R}^{p\times\hat{n}}$ *and* $\hat{D} \in \mathbb{R}^{p\times m}$. Obviously, for $\hat{\Sigma}$ we require $\hat{n} \ll N$ and the error $\|y - \hat{y}\|$ to be small. Depending on the specific norm we choose for the minimization problem, there are different techniques that have been proven to be very successful. On the one hand, there are interpolation-based model reduction techniques that try

to minimize the error in the $\mathcal{H}_2$-norm and, on the other hand, methods like balanced trunction focus on a small $\mathcal{H}_2$-error of the reduced-order system.

The question that immediately arises is how to construct $\hat{\Sigma}$, given an original system Σ. As it turns out, a reduced-order system can be obtained by a projection-type framework. For this, we briefly state the most important properties of projection matrices, see, e.g., [41,42]

Projective Model Order Reduction is based on the idea that normally the state trajectory *x(t)* does not transit all parts of the state space equally often, but mostly constrains to remain inside a subspace V of lower dimension.

The projection theory demonstrates that a projector ***P*** may be defined by using two subspaces ***M*** and ***L*** such that :

$$\boldsymbol{M} \cap \boldsymbol{L}^{\perp} = \boldsymbol{0} \quad \text{(II.2)}$$

Where ***M*** is the subspace spanned by the range ***P*** and ***L*** is the subspace orthogonal to the null subspace of ***P***. The projector ***P*** can define the projection $\hat{x}$ of the vector *x* over the subspace ***M*** and the error vector *e* over the subspace $\boldsymbol{L}^{\perp}$ as follows

$$\hat{x} = Px \quad \text{(II.3)}$$

$$e(x) = x - Px \quad \text{(II.4)}$$

Where $e(x) \in Ran(I - P)$. It is useful to associate the error with an *m*-dimensional subspace V_m which is orthogonal to $Ran(I - P)$. that is $e(x) \perp W_m$. a projection is called an orthogonal projection if W_m is equal to V_m otherwise it is called oblique projection. Figure II.2 illustrates the two types of projection.

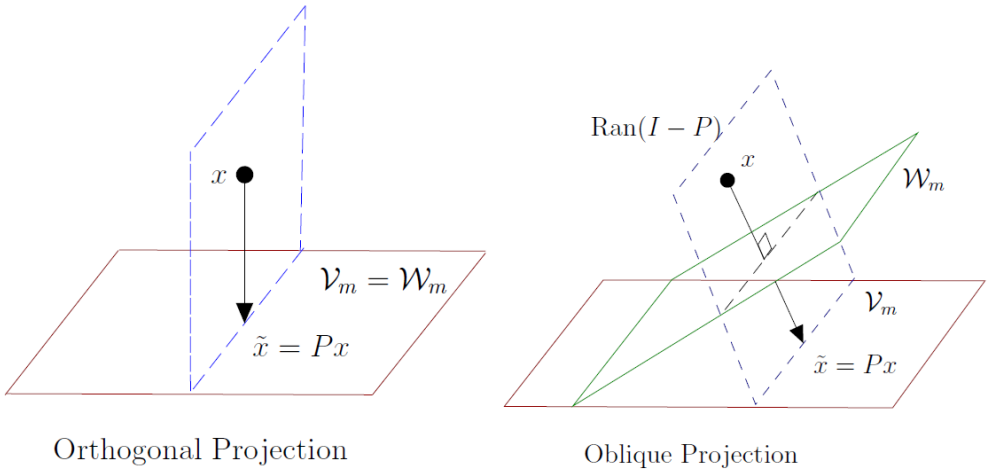


Figure 2. 2 Orthogonal and Oblique Projections

Now for the LTI system given by (I.1), we have, the matrices ***V*** being basis for the subspace ***M*** and ***W*** basis for the subspace ***L*** , the projection technique find an approximate solution ***z*** that is element of the subspace ***M*** and an error vector ***e*** orthogonal to the subspace ***L*** (Pterov-Galerki condition) as follows [43,44]

$$x = Vz \quad \text{(II.5)}$$

The error of the system is then defined as

$$e = b + (A - sE)x \quad \text{(II.6)}$$

$$W^T e = W^T b + W^T (A - sE) V z = 0 \quad \text{(II.7)}$$

The matrices of the system projected on ***M*** and the orthogonal ***L*** are

$$\hat{E} = W^T E V, \quad \hat{A} = W^T A V, \; \hat{B} = W^T B, \; \hat{C} = CV, \; \hat{D} = D \quad \text{(II.8)}$$

II.3.2 Error Model and Error Norms

Every reduced order model (ROM) is associated with a corresponding error system:

Definition 2.1 Given an original system *G(s)* and some reduced order model $G_r(s)$, we define the associated error model $G_e(s)$ as $G_e(s) = G(s) - G_r(s)$.

It is of high importance for the analysis of the approximation quality, which is typically judged by means of $\|G_e\|$, where $\| \cdot \|$ denotes a system norm of interest.

We therefore define the absolute and relative error norms.

Definition 2.2 the absolute $\mathcal{H}_2$ and $\mathcal{H}_\infty$ error norms is defined as

$$\epsilon_{\mathcal{H}_2} = \|G_e\|_{\mathcal{H}_2} \quad and \quad \epsilon_{\mathcal{H}_\infty} = \|G_e\|_{\mathcal{H}_\infty} \qquad \text{(II.9)}$$

Respectively. The relative error norms are defined as

$$\epsilon_{\mathcal{H}_{2,rel}} = \frac{\|G_e\|_{\mathcal{H}_2}}{\|G\|_{\mathcal{H}_2}} \quad and \quad \epsilon_{\mathcal{H}_{\infty,rel}} = \frac{\|G_e\|_{\mathcal{H}_\infty}}{\|G\|_{\mathcal{H}_\infty}} \qquad \text{(II.10)}$$

Projection methods are extremely popular with the area of model order reduction and for both classes (SVD and Krylov techniques). Most of these methods are developed for linear problems, and researchers in recent years have concentrated on aspects like passivity, parametrization and structure preservation[45,46].

II.4 SVD-based MOR techniques

SVD-based model reduction is a well-known system-theoretic approach. The main idea of these methods is to identify the states, which are *important* and *less important* with respect to the input-output behavior of the system. A less important state can be defined as a state, which is hard to control as well as hard to observe. In other words, a less important state requires a lot *input energy* to reach and yet produces very little *output energy*. In the following, we provide a brief review on two well-known techniques for this class of MOR.

II.4.1 Balanced truncation based model order reduction

The main idea of the model order reduction based on balanced truncation is to rearrange the system in its balanced realization form. This means that the states of the system are arranged such that, the more controllable and observable modes of the system are at the beginning of the system's realization. Consequently, the original system can be truncated where its modes become irrelevant for representing its behavior.

Borrowing the notation from [47], given a system of the new form

$$\dot{x} = Ax + Bu, \;\; y = Cx. \qquad \text{(II.11)}$$

A system is in its balanced realization form if its *controllability* W_c and *observability* W_o Grammians are equal; however, most likely, this is not the case of the original system (II.11).

Thus, the balanced realization theory establishes that there exists a similarity transformation $\boldsymbol{x=Tz}$ such that the condition above is met. Moreover, the eigenvalues of the product W_cW_o does not change over any coordinate change[48]. This means that these values, called Hankel values, have the same meaning for the original and transformed system.

The problem of balanced truncation model reduction is narrowed to the calculation of the matrix T, and this calculation can be carried out as follows [27].

The controllability and observability Grammians can be computed from the Lyapunov equation (II.12) and (II.13) respectively.

$$AW_c + W_c A^T + BB^T = 0, \quad \text{(II.12)}$$

$$A^T W_o + W_o A + C^T C = 0, \quad \text{(II.13)}$$

The Cholesky factorization of W_c and W_o are computed by

$$W_c = L_c L_c^T, \quad \text{(II.14)}$$

$$W_o = L_o^T L_0 \quad \text{(II.15)}$$

The singular value decomposition of the product $L_o^T L_c$ is calculated.

$$L_o^T L_c = U\Sigma V^T \quad \text{(II.16)}$$

Where Σ is a diagonal matrix containing the invariant Hankel values of the system. Finally, the matrix T can be computed as

$$T = L_c^T U \Sigma^{-\frac{1}{2}} \quad \text{(II.17)}$$

The balanced realization of the system (II.11) can be obtained by projection:

$$\hat{A} = T^{-1}AT, \ \hat{B} = T^{-1}B, \ \hat{C} = CT \quad \text{(II.18)}$$

The matrices of the balanced system above can be decomposed as follows

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \ \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ \hat{C} = [C_1 \quad C_2] \quad \text{(II.19)}$$

Where the first blocks of the matrices are related with the first r^{th} more controllable and observable states. The reduced system can be established by extracting the matrices from (II.19) or by projecting the original system over the first r^{th} columns of the matrix T.

$$A_r = T_r^T A T_r, \ B_r = T_r B, \ C_r = C T_r^T \quad \text{(II.20)}$$

The balanced truncation method preserves the original modes of the system that have not been truncated; therefore, the reduced model present a good representation of the original system in a wide variety of frequencies[47,49]. However, the calculation of Σ and the two Lyapunov is computationally demanding since it requires computing the singular value decomposition of the system, and is even more difficult for high order systems[47] as the examples cited in the previous chapter.

II.4.2 Modal approximation technique

The modal approximation technique is probably one of the first model reduction technique developed for linear systems. Due to its simplicity, it is still popular for some applications. Here we will explain the main idea of modal approximation. Let

$$G(s) = C(sI - A)^{-1}B \qquad \text{(II.21)}$$

Be the transfer function of (II.11). One can rewrite the transfer function as a partial fraction expansion:

$$G(s) = \sum_{i=1}^{n} \frac{R_i}{s-\lambda_i} \quad \text{(II.22)}$$

A general framework for modal approximation of the transfer function (II.22) is the following [50,51]

1. Compute the poles λ_i and corresponding left and right eigenvalues y_i and x_i;
2. Sort (λ_i, R_i) in decreasing $\frac{|R_i|}{|Re(\lambda_i)|}$ order;
3. Truncate at $\frac{|R_i|}{|Re(\lambda_i)|} < R_{min}$
4. Construct

$$G_k(s) = \sum_{i=1}^{k} \frac{R_i}{s-\lambda_i} \quad \text{(II.22)}$$

Thus, the idea behind the modal approximation is to take the part of the transfer function with the poles that are the closest to the imaginary axis and to throw away the others.

If *n* is large, the step 1 of this framework is in general not feasible since full space eigen-methods such QR and QZ [52] have time complexity $O(n^3)$. In practical situations an alternative for step 1, is the Subspace Accelerated DPA (SADPA) [53,54]. SADPA computes k ($k \ll n$) most

dominant poles and corresponding eigenvectors in an iterative way. The accuracy of the approximation can be controlled by stopping convergence to additional poles when the relative error $\frac{\|G(i\omega_i)-G_k(i\omega_i)\|}{\|G(i\omega_i)\|}$ (measured over several frequencies ω_i) is smaller than a specified tolerance *tol* [55]. An advantage of the modal approximation is that the most dominant poles and residues are exactly those of the original system and, therefore, stability is preserved.

II.5 Krylov sub-space based MOR techniques

It is not in the scope of this thesis to provide extensive details on LTI model reduction methods that exist in the literature. However, to classify the results presented in this thesis it is helpful to point out the practical problems that are related to the goal of this work, which is: model order reduction using a Krylov sub-space technique, and to mention some principle components of this technique such as the concept of moment matching and a description of the Arnoldi Algorithm.

II.5.1 Krylov sub-space and moment matching

Krylov subspace model order reduction techniques are based on the projection onto Krylov subspaces. Due to their property of matching moments of the transfer function of the reduced and original system, Krylov subspace techniques are also called moment matching techniques [33]. The moments of a system are defined as the coefficients of the Taylor series expansion of the transfer function [34-36]. The moments of the system (I.5) around interpolation point $s = 0$ are:

$$M_i = C(A^{-1}E)^i A^{-1}B \quad for \quad i = 0,1 \ldots \qquad \text{(II.8)}$$

The moments off the system around $s = 0$ can be approximated from the Krylov subspace by using the appropriate initial vectors. This is desired since using a more mathematically stable procedure (Arnoldi Algorithm) based on the Krylov subspace instead of the using the explicit Taylor series elements. Before continuing it is appropriate to give the definition of Krylov subspace as follow.

Definition 2.1 A i^{th} dimensional Krylov subspace corresponding to some matrix A and vector g is denoted $K_i\,(A,g)$ and is defined as

$$K_i\,(A,g) = span\left\{g, Ag, A^2g, \ldots, A^{i-1}g\right\} \quad \text{(II.9)}$$

The goal of Krylov subspace model order reduction is to find a projection-based approximation of the original transfer function, which matches the first *k* moments of the original transfer function. A basis for a Krylov subspace can be quickly computed if *A* can be rapidly applied to *g*. e.g.. due to sparsity. This fact gives Krylov-based model order reduction the potential for cost savings. Let

$$H(s) = C(sI - A)^{-1}B = CF(s) \quad \text{(II.10)}$$

Where *F(s)* is the solution to the linear system of equation *(sI – A)F(s) = B.* The krylov projection technique gives an *m*-dimensional approximate solution to this linear system and thus an approximation to the transfer function *H(s).* This approximation is called a single-sided (one-sided) projection. Note that (II.10) can also be written as,

$$H(s) = X(sI - A)^{-1}B = G(s)^T(sI - A)F(s) \quad \text{(II.11)}$$

Where *G(s)* is solution to the linear system, $(sI - A)^T G(s) = C^T$. Now we have two linear systems of equations and the system *H(s)* can be approximated by using Krylov projection technique for each linear system. The resulting approximation is called a two-sided projection.

II.5.1.1 Moment Matching

Generally speaking, moment matching can refer to any projection technique wherein projections are constructed in a way to ensure that the reduced model matches some number of values and derivatives of the transfer function *H(s)* of the original model at a prescribed set of frequencies. We derive formulas for the higher-order derivatives of *H(s).* Based on these results, we define the so called moments of the transfer function, which will be used in the further sections of the approximation accuracy of the reduced-order model [37].

Theorem 2.1 Let *H(s)* be a transfer function as in (I.5), i.e. $H(s) = C^T(sE - A)^{-1}B + D$. Then for all integer $i>0$ [37].

$$\partial^i H(s) = (-1)^{-1} i! \, . C^T((sE - A)^{-1}E)^i(sE - A)^{-1}B = (-1)^i i! . C^T(sE - A)^{-1}(E(sE - A)^{-1})^i B \quad \text{(II.12)}$$

And based on the Taylor series expansion around $s_0 \in \mathbb{C}$

$$H(s) = D + \sum_{i=0}^{\infty} \frac{(s-s_0)^i}{i!} \partial^i H(s_0) = D + \sum_{i=0}^{\infty} (-1)^i (s - s_0)^i C^T((s_0E - A)^{-1}E)^i(s_0E - A)^{-1}B \quad \text{(II.13)}$$

For all s in an open neighborhood of s_0, which for $s_0=0$ and $E=I$ reduces to

$$H(s) = D - \sum_{i=0}^{\infty} s^i C^T A^{-(i+1)} B \quad \text{(II.14)}$$

For all s in an open neighborhood of 0.

Proof. For $i>0$

$$\partial^i H(s) = C^T(\partial^i(sE - A)^{-1})B = (-1)^i i!\, C^T((sE - A)^{-1})E)^i(sE - A)^{-1})B \quad \text{(II.15)}$$

Furthermore

$$H(s) = \sum_{i=0}^{\infty} \frac{(s - s_0)^i}{i!} \partial^i H(s_0)$$

$$= D + \sum_{i=0}^{\infty} \frac{(s - s_0)^i}{i!} (-1)^i i!.\, C^T((s_0E - A)^{-1}E)^i(s_0E - A)^{-1} B$$

$$= D + \sum_{i=0}^{\infty}(s - s_0)^i\, (-1)^i.\, C^T((s_0E - A)^{-1}E)^i(s_0E - A)^{-1} B \quad \text{(II.16)}$$

Which for $s_0=0$ and $E=I$ reduces to

$$H(s) = D + \sum_{i=0}^{\infty}(s - 0)^i\, (-1)^i.\, C^T((0E - A)^{-1}E)^i(0E - A)^{-1} B$$

$$= D + \sum_{i=0}^{\infty}(-s)^i(-1))^{i+1} C^T A^{-(i+1)} E^i B$$

$$= D - \sum_{i=0}^{\infty} s^i C^T A^{-(i+1)} B \quad \text{(II.17)}$$

For all s in an open neighborhood of 0.

Theorem 2.2 (the i^{th} moment of the transfer function $H(s)$); Assuming $D=0$, the i^{th} moment of $H(s)$ at $s_0 \in \mathbb{C}$ is defined as [36]

$$M_i(s_0) = C^T((s_0E - A)^{-1}E)^i(s_0E - A)^{-1}B. \quad \text{(II.18)}$$

II.5.1.2 Moments at infinity

To obtain the Taylor series expansion at infinity, one substitutes 1/*s* for *s* and calculates the Taylor series' coefficient at *s =0*. Let *T(s) = 1/s*. one can not straightforwardly apply the chain-rule to $H_\infty(s) = H(1/s) = (H \circ T)(s)$, since it leads to

$$\partial H_\infty(s) = \partial(H \circ T)(s) = \partial H\big(T(s)\big) \cdot \partial T(s), \quad \text{(II.19)}$$

Where the factor $\partial T(s) = \frac{1}{s^2} \to -\infty \; for \; s \to 0$. Therefore we follow an alternative approach based on a rewritten for of H_∞:

$$H_\infty(s) = H\left(\frac{1}{s}\right) = C^T G\left(\frac{1}{s}\right) B + D = C^T \left(s.\frac{1}{s} G\left(\frac{1}{s}\right)\right) B + D = C^T s.G_\infty(s)B + D, \quad \text{(II.20)}$$

Where $G_\infty(s) = \frac{1}{s} G\left(\frac{1}{s}\right) \; and \; G(s) = (sE - A)^{-1}$; and

$$G_\infty(s) = \frac{1}{s} G\left(\frac{1}{s}\right) = \left(\frac{E}{s} - A\right)^{-1} \frac{1}{s} = \left(\frac{E - sA}{s}\right)^{-1} = ((s(-A) - (-E)))^{-1} \quad \text{(II.21)}$$

We put $F_\infty(s) = (s(-A) - (-E))$ is differentiable for $s \downarrow 0$. This implies that for *I* fixed

$$\partial^i G_\infty(s) = (-1)^i i! \, (G_\infty \partial F_\infty)^i G_\infty = (-1)^i i! \, (G_\infty \circ -A)^i G_\infty = i! \, (G_\infty \circ A)^i G_\infty. \quad \text{(II.22)}$$

Hence for all *i* >0

$$\begin{aligned} \partial^i H_\infty(s) &= \partial^i C^T s G_\infty(s) B + \partial^i D D \\ &= C^T \big[\partial^i \big(s G_\infty(s)\big)\big] B \\ &= C^T \big[i \partial^{i-1} G_\infty(s) + s \, \partial^i G_\infty(s)\big] B \\ &= C^T \big[i(i-1)! \cdot (G_\infty(s)A)^{i-1} G_\infty(s) + s \cdot i! \cdot (G_\infty(s)A)^i G_\infty(s)\big] B \\ &= C^T \big[i! \, ((G_\infty(s)A)^{i-1} G_\infty(s) + s(G_\infty(s)A)^i G_\infty(s))\big] B \quad \text{(II.23)} \end{aligned}$$

Let *i* >*0*. Since $G_\infty(0) = E^{-1}$ one obtains the *derivatives of H at* ∞ :

$$\partial^i H_\infty(0) = i! \cdot C^T (E^{-1}A)^{i-1} E^{-1} B. \quad \text{(II.24)}$$

This shows that the Taylor series at infinity of *H* is that of H_∞ at 0, which is

$$H_\infty(s) = \sum_{i=0}^{\infty} \frac{(s-0)^i}{i!} \cdot \partial^i H_\infty(0)$$

$$= D + \sum_{i=1}^{\infty} \frac{s^i}{i!} \cdot i!\, C^T (E^{-1}A)^{i-1} E^{-1} B$$

$$= D + \sum_{i=1}^{\infty} s^i \cdot C^T (E^{-1}A)^{i-1} E^{-1} B \quad \text{(II.25)}$$

for all *s* in an open neighborhood of 0; i.e.,

$$H(s) = D + \sum_{i=1}^{\infty} s^{-i} \cdot C^T (E^{-1}A)^{i-1} E^{-1} B, \;\; s \to \infty \quad \text{(II.26)}$$

Definition 2.2 (*The i^{th} moment of the transfer function H(s) at infinity)*. Assuming *D =0*, the i^{th} moment of the transfer function *H(s)* at infinity is called a *Markov parameter* and defines as [56]

$$M_{-i} = C^T (E^{-1}A)^{i-1} E^{-1} B. \quad \text{(II.27)}$$

II.5.2 Arnoldi Algorithm

Arnoldi-based and Lanczos-Based techniques are the two commonly used processes to generate the orthonormal basis for the Krylov subspace. The Lanczos process is suitable for two-sided projections since the algorithm computes the orthogonal matrices $\boldsymbol{W}$ and $\boldsymbol{V}$ simultaneously [56]. The Arnoldi-based algorithm can be used twice to generate the basis corresponding to input and output Krylov subspace [57].

In this thesis, we focus on the Arnoldi-like algorithms to reduce an LTI model, so in the next, we describe this type of process. For more details on Lanczos-based algorithm, see [56,58].

As mentioned before, after selecting the suitable vectors that form the Krylov subspace, an orthonormal basis of this subspace should be computed. An Arnoldi-based technique for model order reduction is based on orthogonal projection onto the Krylov subspace, $K_i\,(A, g)$, defined in (II.9). the need for this orthonormal basis underlies inn the fact that the vectors in the Krylov subspace conserve a certain level of dependency [56]. This section describes the computation of one-sided and two-sided Arnoldi approximation.

II.5.2.1 One-sided Arnoldi Algorithm

In the standard form, the one-sided Arnoldi algorithm iteratively computes $V_m \in \mathbb{R}^{Nxm}$ for a given $A \in \mathbb{R}^{NxN}$ and $B \in \mathbb{R}^{N}$, such that the following hold :

- Orthonormal matrix : $V_m{}^T V_m = I_m$

- Basis to Krylov subspace : *colsp (Vm)=$K_m(A,B)$=colps([B AB,…., $A^{m-1}B$])*, where *colps(.)* represent the column span.

The next algorithm (Algorithm 2.1) is a variant of the standard Arnoldi algorithm. The main step of the Arnoldi algorithm is the *Gram-Schmidt* orthogonalization.

Algorithm 2.1 : One-sided Arnoldi Algorithm

Input Matrices: *A*; *B*; *m*; *and tol*

Output Matrices: V_m, $v_{m+1;}$

---

1. **Initialization**
2. $v_1 = B / ||B||_2$
3. $V_1 = []$
4. **for** *i = 1* ***to*** *m* ***Do***
5. $\boldsymbol{V}_i := [V_i\ v_i]$
6. $R_{i+1} := Av_i - V_i\ V^T_i\ Av_i;$
7. *If* $||R_{i+1}|| <$ *tol then stop*
8. $v_{i+1} := R_{i+1} / ||\ R_{i+1}\ ||\ ;$
9. **Next** *j*

II.5.2.2 Two-sided Arnoldi Algorithm

The two-sided Arnoldi approach may increase the accuracy of the Arnoldi approximation [59,60]. Unlike the one-sided Arnoldi method, the two-sided Arnoldi computes two orthogonal basis V_m, and $W_m \in R^{nxm}$ for $K_m(A,B)$ and $K_m(A^T,C^T)$ respectively:

$$colsp(V_m) = colsp([B\ AB, \dots, A^{m-1}B]); \quad V_m^T V_m = I_m \quad \text{(II.19)}$$

$$colsp(W_m) = colsp([C^T\ A^T C^T, \dots, (A^T)^{m-1} C^T]); \quad W_m^T W_m = I_m \quad \text{(II.20)}$$

Thus in the two-sided Arnoldi algorithm, W_m is not arbitrary. A version of two-sided Arnoldi is given in Algorithm 2.2. By construction, it can be shown that the Arnoldi equations:

$$AV_m = V_m A_{Vm} + v_{m+1} C_{Vm}, \quad B = V_m B_{Vm} \quad \text{(II.21)}$$

$$AW_m = W_m A^T_{Wm} + w_{m+1} B^T_{Wm}, \quad C^T = W_m C^T_{Wm} \quad \text{(II.22)}$$

Where $A_{Vm} = V^T_m AV$, $A_{Wm} = W^T_m AW_m$, $B_{Vm} = V^T_m B$, $C_{Vm} = v^T_{m+1} AV_m$, $C_{Wm} = CW_m$, $B_{Wm} = W^T_m Aw_{m+1}$ in which $w_{m+1} \in R^N$ is part of the orthonormal matrix $W_{m+1} = [W_m \; w_{m+1}]$, are satisfied [39]. The above equations can be transformed to represent the oblique projection process.

Algorithm 2.2 : Two-sided Arnoldi Algorithm

Input Matrices: *A*; *B*; C^T, *m*; *and tol*

Output Matrices: V_m, $v_{m+1;}$ W_m, w_{m+1}

---

1. **Initialization**
2. $v_1 = B / ||B||_2$, $w_1 = C^T / ||C^T||$
3. $V_1 = []$, $W_1 = []$
4. **for** $i = 1$ ***to*** m ***Do***
5. $\boldsymbol{V}_i := [V_i \; v_i]$, $W_i = [W_i \; w_i]$
6. $R_{i+1} := Av_i - Vi \; V^T_i \, Av_i;$
7. $P_{i+1} := A^T w_i - W_i W^T_i \, A^T w_i ;$
8. *If* $||R_{i+1}|| <$ *tol or* $||P_{i+1}|| <$ *tol then stop*
9. $v_{i+1} := R_{i+1} / || \, R_{i+1} \, || \, ;$
10. $w_{i+1} := P_{i+1} / || \, P_{i+1} \, || \, ;$
11. **Next** *i*

II.6 Conclusion

Model order reduction approaches are used in a variety of areas. Mechanical systems is one of these areas because the dynamic modelling of mechanical systems often necessitates high-order systems. Model order reduction approaches are classed based on the area in which they are used or their mathematical basis. Several of these approaches, however, operate by projecting the original system onto a smaller subspace. Krylov subspace-based and Balanced truncation are two of these projection-based approaches. The Krylov subspace attempts to reproduce the r^{th} moments of the original system on the moments of the reduced system. The Arnoldi approach is a well-known Krylov-based technique for model order reduction. It can be used to generate an orthonormal basis for the Krylov subspace projection. The commonly used Arnoldi process is the one-sided and two-sided Arnoldi Algorithms.

Chapter Three
Proposed Technique

Chapter III: Proposed Technique

III.1 Introduction

The large-scale 2nd order model is considered a well-known representation for modeling the dynamical behavior of multivariable complex systems in various fields of science and engineering, such as electrical, mechanical, structural, electromagnetic, and micro-electromechanical systems (MEMS). Some of these systems encounter computational problems in simulation due to its huge model order, to treat this problem. Therefore, we desire to find a reliable approximated model with reduced order which replace the original model in the simulation or the control, which can preserve the second order structure of the original system and the same key properties of the full order model such as stability [61,62].

In this thesis, the following large-scale system given in the second-order form is considered, and given by its representation as follows:

$$\sum_N : \begin{cases} \mathbf{M}\ddot{\mathrm{q}}(t) + \mathbf{D}\dot{\mathrm{q}}(t) + \mathbf{K}\mathrm{q}(t) = \mathbf{b}u(t) \\ y(t) = l^T\mathbf{q}(t) \end{cases} \quad \text{(III.1)}$$

$\mathbf{M} \in \mathbb{R}^{N\,x\,N}$ is invertible matrix, $\mathbf{q}(\mathbf{0}) = \mathbf{q_0}$ and $\dot{\mathbf{q}}(\mathbf{0}) = \dot{\mathbf{q}}_\mathbf{0}$ **,** initial conditions. **M, D, and K** $\in \mathbb{R}^{N\,x\,N}$ **are** respectively the mass, damping and stiffness matrices as known for mechanical models. $\mathbf{q}$ $(t) \in \mathbb{R}^N$ is the state variables vector. $\mathbf{b}$, $\mathbf{l} \in \mathbb{R}^N$ represent respectively the input distribution matrix and the output measurement matrix [62].

In this chapter, a model order reduction (MOR) of the 2nd order system $\sum_N$ using a Second Order Arnoldi (SOAR) algorithm was discussed. The SOAR approach has been attracted many researchers in the last few years, which has been used to solve the following problems: a quadratic Eigen-value [63,64], the MOR of second-order dynamical systems [62,65] and in the analysis of structural acoustics [65,66]. From mathematical point of view, the SOAR design is based on a projection-based MOR technique that uses a second-order Krylov subspace and the SOAR procedure to generate the projection matrix as follows: first step a recurrence formula is

defined for the two matrices coefficient A and B and one or two initial vectors, the second step generate an orthonormal basis of projection sub-spaces from the famous second-order Krylov subspace defined in the recurrence formula.

Obviously, the SOAR technique is used in MOR, which construct another reduced second-order state-space system ∑n with reduced order, where the input-output behavior dynamics are completely recovered i.e. preserving the basic characteristics of the full order system [61,62].

In this work, we try to automate the generation of the best reduced order model for a large second order system using the SOAR procedure, by defining a new criterion to auto-stop the iteration process in SOAR procedure and to auto-select an acceptable reduced order of the projection matrix, which in the limit of our knowledge is a new proposed idea, the efficiency and robustness of the proposed algorithm is validated by various well-chosen numerical examples of 2nd order models.

In the next two sections (Section.2 and 3) of this chapter, we give some preliminaries concerning reduction of large-scale second order models based on the SOAR technique. Section.4 describes the theoretical establishment of Numerical Rank Performance Coefficient (NRPC) and proposes a stopping condition based on the NRPC value.

III.2 Model Reduction Of Second Order System

III.2.1 Problem formulation

Numerous approaches have been proposed throughout the years for structural-preserving model reduction of the second-order system [61]. First a structure preserving method with moment-match property was proposed by Su and Craig in 1991 [67]. Improvements have been added by Z. Bai and Y. Su. in 2005-2006 [61,68].

The analysis of the second-order system $\sum_N$ leads to building a very large complex model. Thus, we need mathematical tools for reducing the computational complexity and to accelerate the modeling task by constructing a reduced order system $\sum_n$. Where, the proposed tools preserve the characteristic proprieties of the original system such as, the second order form and stability, the reduced second order system $\sum_n$ is defined by:

$$\sum_n : \begin{cases} \mathbf{M}_n \ddot{\mathrm{z}}(t) + \mathbf{D}_n \dot{\mathrm{z}}(t) + \mathbf{K}_n \mathrm{z}(t) = \mathbf{b_n} u(t) \\ \hat{y}(t) = {l_n}^T \mathbf{z}(t) \end{cases} \quad \text{(III.2)}$$

Where $\mathbf{z}(t)$ represent the state vector with dimension n, where $n << N$. and the matrices $\mathbf{M}_n$, $\mathbf{D}_n$, and $\mathbf{K}_n \in \mathbb{R}^{n\,x\,n}$, are the mass, damping, and stiffness matrices as known in structural dynamics, and the vectors $\mathbf{b}n$, $\mathbf{l}n \in \mathbb{R}^{\,n}$ are input distribution, output measurement.

By applying the Laplace transform, the transfer function of the original second-order model is given as follows:

$$h(s) = l^T (\mathbf{M}s^2 + \mathbf{D}s + \mathbf{K})^{-1} b \quad \text{(III.3)}$$

The power series of Laplace transform Q(s) = L (q (t)) can be expressed as follows:

$$Q(s) = R_0 + R_1 s + \ldots = \sum_{l=0}^{\infty} R_l s^l \quad \text{(III.4)}$$

Where R_l are called l^{th} order system moments. One can rewrite the above second order system $\sum_N$ in first-order form by taking the following state vector $x(t) = \begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix}$:

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \ddot{q}(t) \\ \dot{q}(t) \end{bmatrix} - \begin{bmatrix} -D & -K \\ I & 0 \end{bmatrix} \begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 0 & l^T \end{bmatrix} \begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix} \quad \text{(III.5)}$$

The compact representation of the equation (III.5) can be expressed in matrix form as follow

$$C\dot{x}(t) - Gx(t) = Bu(t), \quad y(t) = L^T x(t) \quad \text{(III.6)}$$

Where: $C = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$, $G = \begin{bmatrix} -D & -K \\ I & 0 \end{bmatrix}$, and $B = \begin{bmatrix} b \\ 0 \end{bmatrix}$, $L^T = \begin{bmatrix} 0 & l^T \end{bmatrix}$ are respectively the input and the output of the first-order system representation.

Now we can write the transfer function h(s) as follows:

$$h(s) = L^T (Cs - G)^{-1} B \quad \text{(III.7)}$$

And the power series of the transfer function h (s) as:

$$h(s) = m_0 + m_1 s + \ldots = \sum_{l=0}^{\infty} m_l s^l \quad \text{(III.8)}$$

The design of MOR Krylov subspace techniques is based on the moment matching methods. The principle of its work is to replicate the moments of the original system $\sum_N$ on the moments of the reduced system $\sum_n$.

The first application of MOR via Krylov Subspaces was on the first order systems, but we consider in this study a system described by 2nd order form. Generally, we proceed by transforming the 2nd order form to the 1st order state space type, but the drawback of this process is the lot of original model structures (sparsity, symmetry, orthogonality, etc.) are lost. Hence, many researches in second order reduction technique has recently been proposed with some constructive achievements in preservation of the original properties [69–71] .

III.2.2 SOAR Algorithm for second order systems

A mathematical formulation of Krylov Subspace is used as framework of a subspace projection technique to obtain a reduced system with the moment matching property, in the next, we define a second order Krylov subspace with presentation of SOAR algorithm.

Definition 1: We define the second-*order Krylov subspace as follow:*

$$G_n(A, Z, w) = colspan\{r_0, r_1, \dots, r_{n-1}\} \text{ (III.9)}$$

Where

$$\begin{cases} r_0 = w, \ r_1 = Ar_0 \\ r_i = Ar_{i-1} + Zr_{i-2} \ , \ i = 2,3, \dots \end{cases} \text{(III.10)}$$

Where $A, Z \in \mathbb{R}^{n\,x\,n}$ are square matrices called constant matrices , and $w \in \mathbb{R}^{n\,x\,1}$ is a column vector called the starting vector., Then the sequence r_0, r_1, …., r_{n-1} is known as *second-order Krylov basic blocks* [62].

The subspace $G_n(A,Z;w)$ defined in (III.9) is called an n^{th} *second order Krylov subspace* [61,68,71]. There is a connection between the subspace $G_n(A,Z;w)$ and a the standard Krylov subspace K_n (A,w) defined in (III.11), we can consider $G_n(A,Z;w)$ as a generalization of K_n (A,w).

$$K_n \ (A,w) = \text{span}\ (w, Aw, A^2w, \dots, A^{n-1}w) \quad \text{(III.11)}$$

In the case of the matrix Z= 0, the second order Krylov subspace $G_n(A,Z;w)$ is equal to the general Krylov subspace $K_n(A,w)$.

The application of the vector sequence $\{r_j\}$ of the second-order Krylov sub-space to obtain an orthonormal basis $\{r_0, r_1, \ldots, r_{n-1}\}$ is called a SOAR (Second Order Arnoldi) Algorithm, this is an Arnoldi-like procedure.

Algorithm 3.1: Second Order Arnoldi Algorithm

Input Matrices: *A*; *Z*; *w*; *q*

Output Matrices: Q_n

---

10. **Initialization**
11. $q_1 = w / ||w||_2$
12. $p_0 = 0$
13. **for** $j = 1$ ***to*** q***Do***
14. $r := Aq_j + Zp_j$
15. $s := q_j$
16. **for** $i = 1$***to*** j***Do***
17. $t_{ij} := q_i^T r$
18. $r := r - q_i t_{ij}$
19. $s := s - p_i t_{ij}$
20. **Next** i
21. $t_{j+1,j} := ||r||_2$
22. **if** $t_{j+1,j} == 0$, **Break**
23. $q_{j+1} := r / t_{j+1,j}$
24. $p_{j+1} := s / t_{j+1,j}$
25. **Next** j

Two loops exist in SOAR algorithm. The first is the inner loop, which is an orthogonalization procedure for $\{q_i\}$ vectors. Where, a Gram-Schmidt procedure is used to generate the orthogonal vectors. The $\{p_i\}$ vectors set is an intermediate sequence. Another modified SOAR Algorithm version was proposed in [61] to avoid the use of a

direct reference to the intermediate vector set of $\{p_i\}$. That will almost halve the size of the needed memory capacity. The second is the outer For-loop which is mainly used for counting the total number of iteration and generating the final orthonormal basis Q_n for the given Krylov subspace.

If the matrices Q_n ,P_n, denote square matrices with dimension n and whose columns vectors $\{q_1 ,q_2 , q_3, \ldots , q_n\}$ and $\{p_1 ,p_2 , p_3, \ldots , p_n\}$, respectively. Let T_n represents the square upper Hessenberg matrix with entries $t_{ij} \neq 0$ obtained from the previous Algorithm.1. Just as the relation in [10], then we easily see that the tow next relations for Q_n, P_n, and T_n holds:

$$AQ_n + ZP_n = Q_nT_n + q_{n+1}e_n^T t_{n+1,n} \quad \text{(III.12)}$$

$$Q_n = P_nT_n + p_{n+1}e_n^T e_{n+1,n} \quad \text{(III.13)}$$

If upper Hessenberg matrix$\tilde{T}_n$with dimension of (n + 1) x n can be written in the form of $\tilde{T}_n = \begin{bmatrix} T_n \\ e^T t_{n+1,n} \end{bmatrix}$, then the Eqs. (III.12) and (III.13) can be rewritten in the compact form

$$\begin{bmatrix} A & Z \\ I & 0 \end{bmatrix} \begin{bmatrix} Q_n \\ P_n \end{bmatrix} = \begin{bmatrix} Q_{n+1} \\ P_{n+1} \end{bmatrix} \tilde{T}_n \quad \text{(III.14)}$$

The sub-space spanned by the vectors $\{q_1 ,q_2 , q_3, \ldots , q_n\}$ generated from the SOAR procedure forms the projection sub-space to produce the reduced model, in other words they form an orthogonal projection matrix by means of Second Order Krylov Subspace.

$$\text{span } \{q_1 ,q_2 , q_3, \ldots , q_n\} = G_n(A,Z;w), \text{ for } n \geq 1.$$

And $q_i^T q_k = \delta_{ik}$ for I, k= 1,…., n [63]. In fact, in moment matching by means of Second Order Krylov Subspaces we use the output of the Hessenberg matrix T_n of the SOAR Procedure to extracted analytically the system moments. The next Lemma.1can be used in the proof of the Theorem.1:

Lemma.1 Let $T_n \in R^{n \times n}$ bean upper Hessenberg matrix. The k entry $T_n^j e_1$is zero for $k = j+2, \ldots, n$ and $j=0,1,..,$n-1. In particular, $e_n^T T_n^j e_1 = 0$ for j = 0, 1, . . ., n-2. For proof see [62]

Theorem.1 Let Q_n be the orthonormal matrix defined by the sequence vectors $Q_n = \{q_1, q_2, \ldots, q_n\}$, where the vectors q_i with i=1, …, n are generated by the SOAR procedure after the

execution of *n* iterations. Then the analysis of the relationship between the j[th] system's moment r_j and the output the SOAR procedure can be expressed as follow: [62]

$$\begin{bmatrix} r_j \\ r_{j-1} \end{bmatrix} = H^j v = \begin{bmatrix} Q_n \\ P_n \end{bmatrix} T_n^j e_1, \quad \text{for } j = 0,1, \ldots, N\text{-}1. \qquad \text{(III.15)}$$

In particular, we have

$$r_j = Q_n T_n^j e_1, \qquad and \; r_{j-1} = P_n T_n^j e_1, \qquad for \; j = 0,1, \ldots, N-1$$

Proof. We can re-write the SOAR algorithm iteration as

$$\begin{bmatrix} A & Z \\ I & 0 \end{bmatrix} \begin{bmatrix} Q_n \\ P_n \end{bmatrix} = \begin{bmatrix} Q_n \\ P_n \end{bmatrix} T_n + \begin{bmatrix} q_{n+1} \\ p_{n+1} \end{bmatrix} t_{n+1,n} e_n^T. \qquad \text{(III.16)}$$

Therefore, if we multiply on the right the both sides of the Eqs. (III.13) by e_1, we have

$$\begin{bmatrix} r_1 \\ r_0 \end{bmatrix} = Hv = \begin{bmatrix} Q_n \\ P_n \end{bmatrix} T_n e_1. \qquad \text{(III.17)}$$

Therefore, $r_1 = Q_n T_n e_1$ and $r_0 = P_n T_n e_1$. The extension of the above argument into the j^{th} iteration for j=0,1,..,n-*1* can be verified mathematically by induction proof. We now admit that Eqs. (III.15) is true for $j = p$, for $j = p+1$, we have

$$\begin{bmatrix} A & Z \\ I & 0 \end{bmatrix}^{p+1} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix} = \begin{bmatrix} A & Z \\ I & 0 \end{bmatrix} \begin{bmatrix} Q_n \\ P_n \end{bmatrix} T_n^p e_1 = \left\{ \begin{bmatrix} Q_n \\ P_n \end{bmatrix} T_n + \begin{bmatrix} q_{n+1} \\ p_{n+1} \end{bmatrix} t_{n+1,n} e_n^T \right\} T_n^p e_1 = \begin{bmatrix} Q_n \\ P_n \end{bmatrix} T_n^{p+1} e_1.$$

(3.18)

Here we're using the fact mentioned before in Lemma 1 $e_n^T T_n^j e_1 = 0$ for j = 0, 1. . . n-2 in the last equality. Therefore, we can say that Eqs. (III.14) defines compactly the moment's representation of the system.

III.2.3 Second order model reduction based on moment matching

A reduced second-order system is generated using the projection approaches, which are subjected for the linear systems, where these projection techniques closes the second-order into the first order system. Let $Q_n \in R^{N \times n}$ such that

$$q(t) \approx Q_n \, z(t) \qquad \text{(III.19)}$$

where *z(t)* is the reduced state vector so that can accurately approximate the high-dimensional vector *q(t)* by the projection Q_n. in other words Q_n is the projection matrix which mapping the *N*-dimensional state space into a reduced *n*-dimensional space. Substituting equation (III.16) into equation (3.1) and by multiplying the first equation of (III.1) with Q_n^T from the left and Q_n from the right yield the reduced system (3.1) where the matrices $\mathbf{M_n D_n}$, $\mathbf{K_n} \in \mathbb{R}^{n\,x\,n}$ are defined as $\mathbf{M_n}=\mathbf{Q_n^T M Q_n}$, $\mathbf{D_n} = \mathbf{Q_n^T D Q_n}$, and $\mathbf{K_n}=\mathbf{Q_n^T K Q_n}$. and the vectors $\mathbf{b_n}$, $\mathbf{l_n} \in \mathbb{R}^n$ are defined as $\mathbf{b_n}=\mathbf{Q_n^T b}$ and $\mathbf{l_n}=\mathbf{Q_n^T l}$. From the explicit formulation of the reduced order model $\sum_n$ matrices $\mathbf{M_n}$, $\mathbf{D_n}$ and $\mathbf{K_n}$, we can notice that the essential structures of the original model matrices **M,D** and **K** are preserved, such as the symmetric positive definite structure of **M**, which is preserved in $\mathbf{M_n}$. Consequently, and as result we can easily say that the stability is also preserved in the reduced Model [61].

In the following theorem.2, we give a sufficient condition for the *n first* output vectors moment of the reduced and the original systems are matched. By using the second-order form.

Theorem.2 Let Q_n be the projection matrix generated by the SOAR algorithm to reduce the order of the second order model. If $r_i \in$ colspan(Q_n) for j =0, …, n then $r_j = Q_n \hat{r}_j$. [62], for proof see [62]

The previous theorem 2. Can helps us now to verify that the condition of the output moment matching can be maintained.

Theorem.3 With assumption of the condition stated in theorem.2, the output moments of the transfer function of both the original and the reduced second-order systems are matched and we write:[62]

$$\hat{m}_i = m_i \text{ for } i = 0, 1, \ldots, n.$$

Theorem.4 The first unmatched moment is obtained after the execution of the n^{th} iteration of the SOAR procedure, and the error Δm_{n+1}between the two moments Δm_{n+1} of the original system and $\hat{m}_{n+1}$ of the reduced system can be given by analytical expression. For proof see [61,62]

$$\Delta m_{n+1} = m_{n+1} - \hat{m}_{n+1} = C^T \left(\prod_{j=1}^{n} t_{j+1,j}\right) q_{n+1} \quad \text{(III.20)}$$

Generally, the SOAR algorithm converges in the case of the equality in dimensions of the original second order system and the reduced system, or when encounter a breakdown in the iteration process [68]. In theorem.4, the decrease of the unmatched moments error cannot be

ensured to be monotonically. To analyze the efficiency of the SOAR procedure in realistic manner we check the converge of SOAR algorithm if the theoretical expression of an unmatched moments error is necessarily small enough, for more details see [72].

III.2.4 SOAR procedure with deflation and memory saving

In algorithm 1, the p_n is closely related to q_n, and the bi-product vector p_n are clearly utilized. So to avoid the explicit references and updates of the p vectors, a new version of SOAR shown in algorithm 2, was presented in [50] to reduce memory requirement by almost half.

Since $p_1 = 0$, we have

$$Q_n = P_{n+1}\hat{T}_n = P_{n+1}(:,2:n+1)\hat{T}_n(2:n+1,1:n). \quad \text{(III.21)}$$

Algorithm 3.2: SOAR with deflation and saving memory Algorithm

Input Matrices :A, Z, r_0, q

Output Matrices: Q_n

---

1. **/* Initialization */**
2. $q_1 := r_0 / ||r_0||_2$
3. $f := 0$
4. **for** $j := 1,2, \ldots, q$
5. $r := Aq_j + Zf$
6. **for** $i := 1, 2, \ldots, j$
7. $t_{ij} := q_i^T r$
8. $r := r - q_i t_{ij}$
9. **Next** i
10. $t_{j+1} := ||r||_2$
11. **if** $t_{j+1,j} != 0$
12. $q_{j+1} := r / t_{j+1,j}$ ***/* Normalization*/***
13. $f := Q_j \hat{T}(2:j+1,1:j)^{-1} e_j$
14. **else**
15. reset $t_{j+1,j} := 1$
16. $q_{j+1} := 0$
17. $f := Q_j \hat{T}(2:j+1,1:j)^{-1} e_j$

18. save *f* and check deflation and breakdown
19. **End-if**
20. **Next** *j*

The purpose of the checking of deflation in line 19 is to verify if the subspace defined by the set of previously saved ***f*** vectors contains the new ***f*** generated at line18 – the modified Gram-Schmidt procedure can be used to check whether ***f*** is a component of this subspace -, in case of ***f*** linearly dependent on all previously saved ***f*** vectors, then we get a breakdown and the algorithm is terminated. Otherwise, we have a deflation at step *j*; and the algorithm set the value of $t_{j+1,j}$ to 1 and continue running [73].

III.3 Proposed Stopping Criterion For SOAR

Finding a suitable order, q, for the reduced model that leads to a better approximation is one of the important components in order reduction. Roughly speaking, the question is: When can we pause in iterative order reduction? [74]. The traditional tow techniques to stop an iterative MOR approach based on Krylov Subspace are:

A. *Finding the zero vector*:

Although it has an automatic implementation, the conventional approach consists in finding the zero vector (t_{j+1}=0 in the SOAR procedure) to interrupt the process is extremely ineffective, the major drawback of this technique is a lot of redundancy is added to thetransformation matrix and the duplication once and again of the same information about dynamic behavior of the original model.

B. *Time response comparison and manual termination:*

In this technique a reduced order model is generated for each iteration of the SOAR procedure, a time response comparison is made for both models the reduced and the original one, and the process is terminated if the errors are acceptable. Although this stopping approach returns reduced iterations of appropriate sizes, a great deal of cost saving in term of computational effort is involved and manual operation is required.

In order to present the implementation and application of the new criterion for stopping and selection of the reduced order in the SOAR procedure, some fundamentals definitions from Matrix Factorization will be presented first [75].

Definition 1. Let $A \in \mathbb{C}^{n \times m}$, and we suppose *rank(A)=r,* and $n \leq m$, then there exist matrices $U \in \mathbb{C}^{n \times n}$ and $V \in \mathbb{C}^{m \times m}$ such and $\sum \in R^{n \times n}$ such that

$$A = U \sum V^{*} \quad \text{(III.22)}$$

U and V are unitary, and $\sum = \begin{bmatrix} \widehat{\sum} & 0 \\ 0 & 0 \end{bmatrix}$

Where $\widehat{\sum}$ *is a diagonal* matrix with $\widehat{\sum} = diag\{\sigma_1 \ldots \sigma_r\}$ with $\sigma_1 \geq \sigma_i \geq \ldots\ldots \geq \sigma_r > 0$.

The positive numbers σ_i for $i = 1 \ldots r$, are determined uniquely by *A* and are called the *singular values* of A. The equation III.22 is called the Singular Value Decomposition (SVD) of matrix *A*; the columns of *U* and *V* are called the left and right singular vectors, respectively. The index *r* of the smallest singular value is called the theoretical rank of matrix *A.*

Definition 2. Let σ_r the calculated singular values of the matrix $A \in \mathbb{C}^{n \times m}$, and let δ a positive real number, $\delta > 0$, we define the numerical δ- rank as the number of the singular values that are greater than δ, we write numerical δ- rank=*k*, if :

$$\sigma_1 \geq \sigma_2 \geq \ldots\ldots \ \sigma_k \geq \delta \geq \sigma_{k+1} \geq \ldots\ldots \geq \sigma_r; \ r = min\ (n, m) \quad \text{(III.23)}$$

For each iteration of SOAR algorithm. The contribution of second order Krylov vectors with taking into consideration the knowledge about the model dynamics stored in them decreases monotonously.it is therefore expected that each generated vector r_i will be less effective to the numerical-rank of the transformation matrix Q_n.

The suggested stopping criteria is relied on the estimation of a signal of the progress in the numerical-rank of the generated transformation matrix with each iteration in SOAR procedure, exactly before adding the new normalized vector.

In order to measure this signal, for each vector generated by the iterative process we assign an indicator before being fed to the normalization routine (line 12 in algorithm 2), we call it NRPC, the value range of NRPC is]0,1]. The greater value of the NRPC for nominee vector harmonize to an important contribution of that vector to the improvement of the numerical rank and to the dynamic progression of the original system and must hence be included in Q_n. The NRPC for the candidate vector r_i is given as the inverse of the sum of singular values of the current

transformation matrix Q_l obtained by appending the new no-normalized vector r to transformation matrix Q_n of the previous iteration.

$$NRPC = \frac{1}{\sum_{j=1}^{i} \sigma_j} \qquad \text{(III.24)}$$

Therefore, it is possible to approximate the stopping criteria as follows:

1. In each iteration calculate the NRPC indicator using Eqs III.24
2. The iteration can be halted as soon as NRPC < ε, where ε > 0 some specified value.

For the sake of example, In Algorithm 3.3, we demonstrate how to integrate this stopping criterion into the Algorithm 3.2 of SOAR.

Algorithm 3.3: SOAR with stopping criterion Algorithm

Input Matrices :A, Z, r_0;q

Output Matrices: Q_n

---

1. **/* Initialization */**
2. $q_1 := r_0 / ||r_0||_2$
3. $f := 0$
4. **for** $j := 1,2, \ldots, q$
5. $r := Aq_j + Zf$
6. **for** $i := 1, 2, \ldots, j$
7. $t_{ij} := q_i^T r$
8. $r := r - q_i t_{ij}$
9. **Next** i
10. $t_{j+1} := ||r||_2$
11. $Q_1 := [Q_n \ r]$
12. $Sigma := svd(Q_1)$
13. $NRPC := 1/sum\ (sigma)$
14. **If** $NRPC < \varepsilon$ **then Stop**
15. **end if**
16. **if** $t_{j+1,j} != 0$
17. $q_{j+1} := r / t_{j+1,j}$ ***/* Normalization***

18. $f := Q_j \hat{T}(2: j+1, 1: j)^{-1} e_j$
19. **else**
20. reset $t_{j+1,j}$:= 1
21. q_{j+1}:=0
22. $f = Q_j \hat{T}(2: j+1, 1: j)^{-1} e_j$
23. save *f* and check deflation and breakdown
24. **End-if**
25. **Next** *j*

III.4 Coclusion

In this chapter, we have considered a new approach for efficient early termination and automatic optimal order selection of the reduced model for a large second-order system. The suggested approach based on a new coefficient, to which we referred as NRPC coefficient, can improve the selection of the reduced order automatically with preservation of the second-order form and some properties of the original model such as stability. In the two first sections of this chapter we present some implementation details of the second-order Krylov sub-space and Second-order Arnoldi Algorithm. In section three, we present the implementation of the NRPC coefficient and its integration in the SOAR approach.

Chapter Four

Numerical Application

Chapter IV: Numerical Application

IV.1 Introduction

In this section, we illustrate the efficiency and the performance of the proposed technique for reducing second-order systems using various examples. We compare the frequency response of the reduced model with the original one for each example. We also check the stability of reduced models by plotting the poles distribution. The Bode diagram and the Step response are used to calculate some norms such as the steady error and to illustrate the stability of the reduced models. To compare the results, we consider different types of the error functions. For relatively small systems, we calculate the relative error defined as:

$$\mathcal{H}_\infty error\_norm = \frac{\|G(s)-G_r(s)\|_\infty}{\|G(s)\|_\infty} \qquad \text{(IV.1)}$$

$$\mathcal{H}_2 error\_norm = \frac{\|G(s)-G_r(s)\|_2}{\|G(s)\|_2} \qquad \text{(IV.2)}$$

Where $\|.\|_\infty$, $\|.\|_2$, $G(s)$ and $G_r(s)$ are $\mathcal{H}_\infty\ norm$, $\mathcal{H}_2\ norm$, the transfer function of the original and reduced systems, respectively.

IV.2 Materials and Methods.

The numerical applications were conducted to demonstrate the efficiency and the accuracy of the SOAR method with the proposed stopping criterion and the robustness of the terminating mechanism for the Krylov based reduction technique, the proposed approach, in this section, is applied to various applications. Two practical engineering examples are studied: (i) A shaft on bearing supports with a damper originated from a Finite Element (FE) in the first application. (ii) The butterfly gyro-scope problem in second application, in order to compare the results of the proposed algorithm for these examples, we consider the output of the relative errors between the original and reduced systems, and also the related frequency responses[47,48].All experiments are implemented in MATLAB 7.12.0 (R2011a) using PC with Intel® Core™ with i5-CPUand 8 GB RAM-Memory.

Table 4.1 displays the dimension of the discussing models, their types with analogous input-output structure, and the chosen values of NRPC and their corresponding size of the reduced order models (ROMs) gain by the developed technique. Detailed of those models are available on the web page for the *Oberwolfach Benchmark Collection.*

Model	Full Order Model (N)	Input/output	NRPC Values	ROMs (n)
Shaft on bearing supports	400	1/1	ε = 0.1, ε = 0.053, ε = 0.026	n=10 n=19 n=40
Butterfly gyroscope	17361	12/12	ε = 0.1, ε = 0.053, ε = 0.026	n=10 n=19 n=39
3D Cantilever Timoshenko beam	600	1/1	ε = 0.1, ε = 0.053, ε = 0.026	n=10 n=19 n=39

Table 4.1 Model examples with input/output structure

IV.3 Pattern of the NRPC Coefficient

Figure 4.1 shows the pattern of the stopping criterion, the NRPC coefficient, which is the inverse of the sum of singular values of the current transformation matrix Q_l obtained by appending the new no-normalized vector r to transformation matrix Q_n of the previous iteration. The NRPC of the transformation matrix Q_n decreases monotonically. The NRPC coefficient decreases rapidly from the value of 1 to 0.05 after 20 iterations, that means, the 20 first vectors generated by the SOAR procedure have dominant contribution in the numerical rank of the transformation matrix. After the 20^{th} iteration the NRPC decreases slowly and we can say that a good reduced order model can be selected around 20 and above, in other terms an acceptable good approximation model can be obtained for values of NRPC in $]0\ ,\ 0.5]$,then we define a fixed tolerance value ε in $]0\ ,\ 0.5]$to implement a conditionfor selecting a good reduced order and stopping the SOAR procedure.

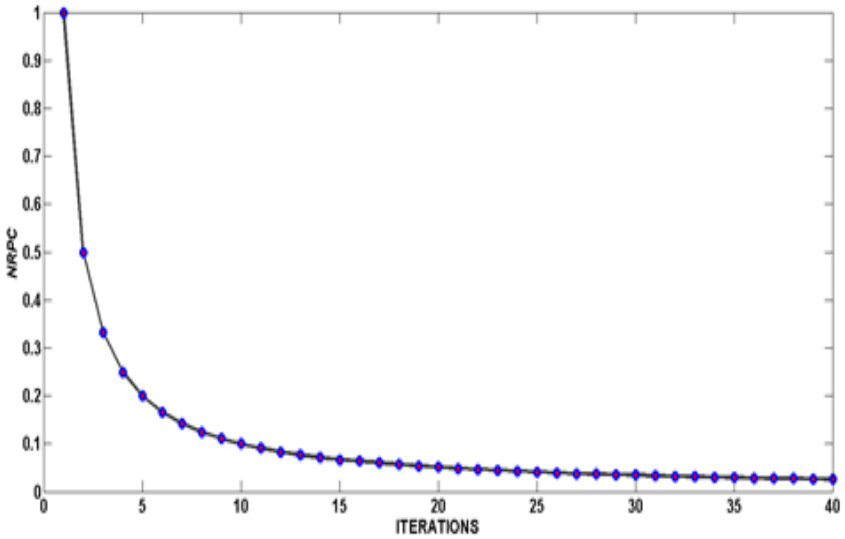


Figure 4. 1 Pattern of variation of NRPC vs number of iterations

IV.4 A shaft on bearing supports

This example comes from Finite Element (FE) model, it is a second order system with full order N= 400, the data of the FE model of a shaft on bearing supports with damper were extracted from MSC-NASTRAN [76]. This second order model is symmetric, with **M** and **D** are symmetric and not positive definite matrices, and **K** is symmetric positive definite matrix. The frequency range of interest of this example is [0, 3000] Hz, and we suggest the use of expansion point s0 = 150×2π. Figure.4.2 shows the frequency response and the relative errors of the exact transfer function *h(s)* and the reduced ones $h_n(s)$ obtained by the proposed method with ε = 0.1, ε = 0.053, ε = 0.026, correspond respectively to reduced models with order n= 10, 19, 40, and Figure.4.3 shows their poles distribution with all negative real parts which means the stability preservation for the reduced models. Figure 4.4 and 4.5, show the Bode diagram and the step response for the reduced order model (n=40) with the proposed method.

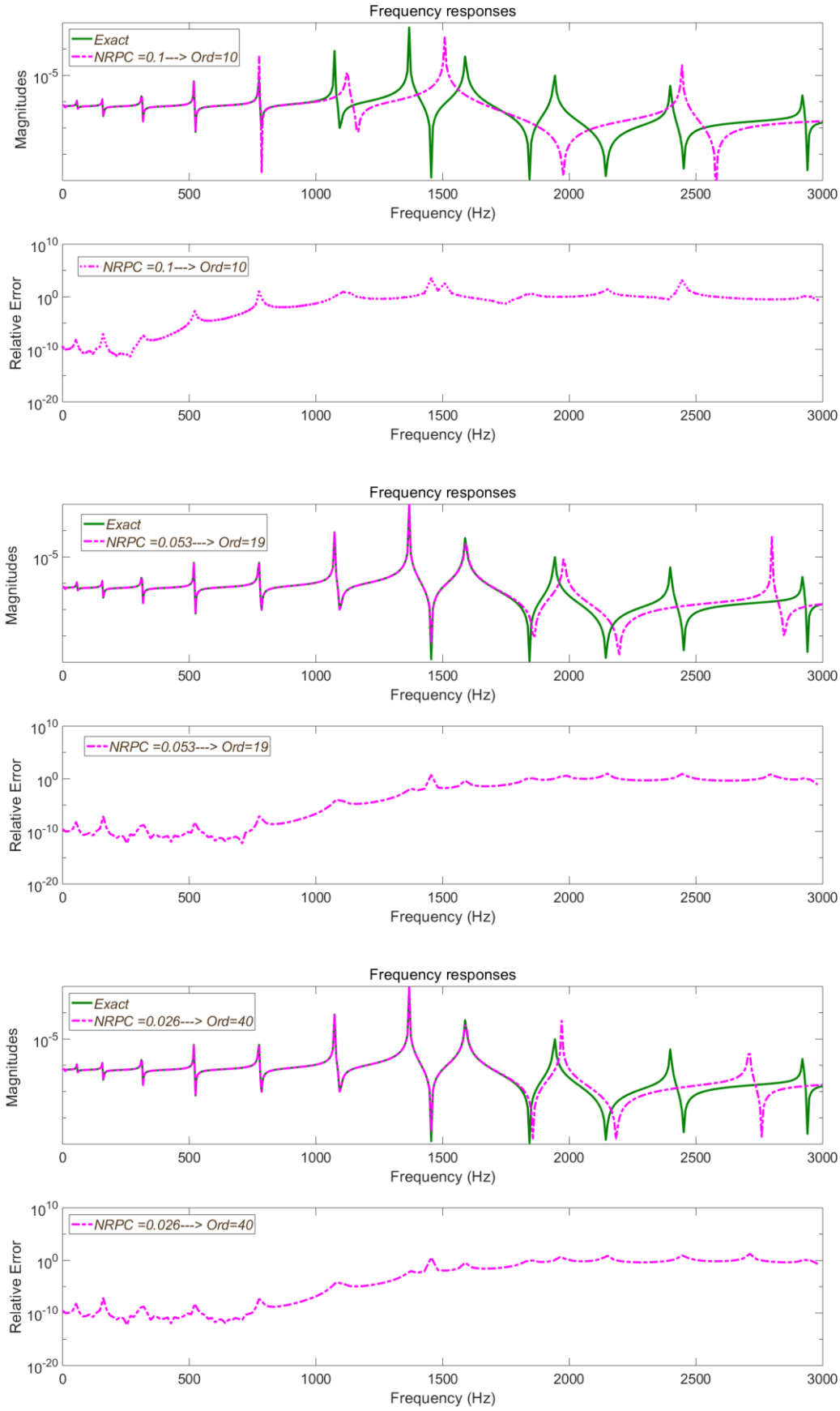


Figure 4.2 Frequency response the relative errors for the shaft on bearing support

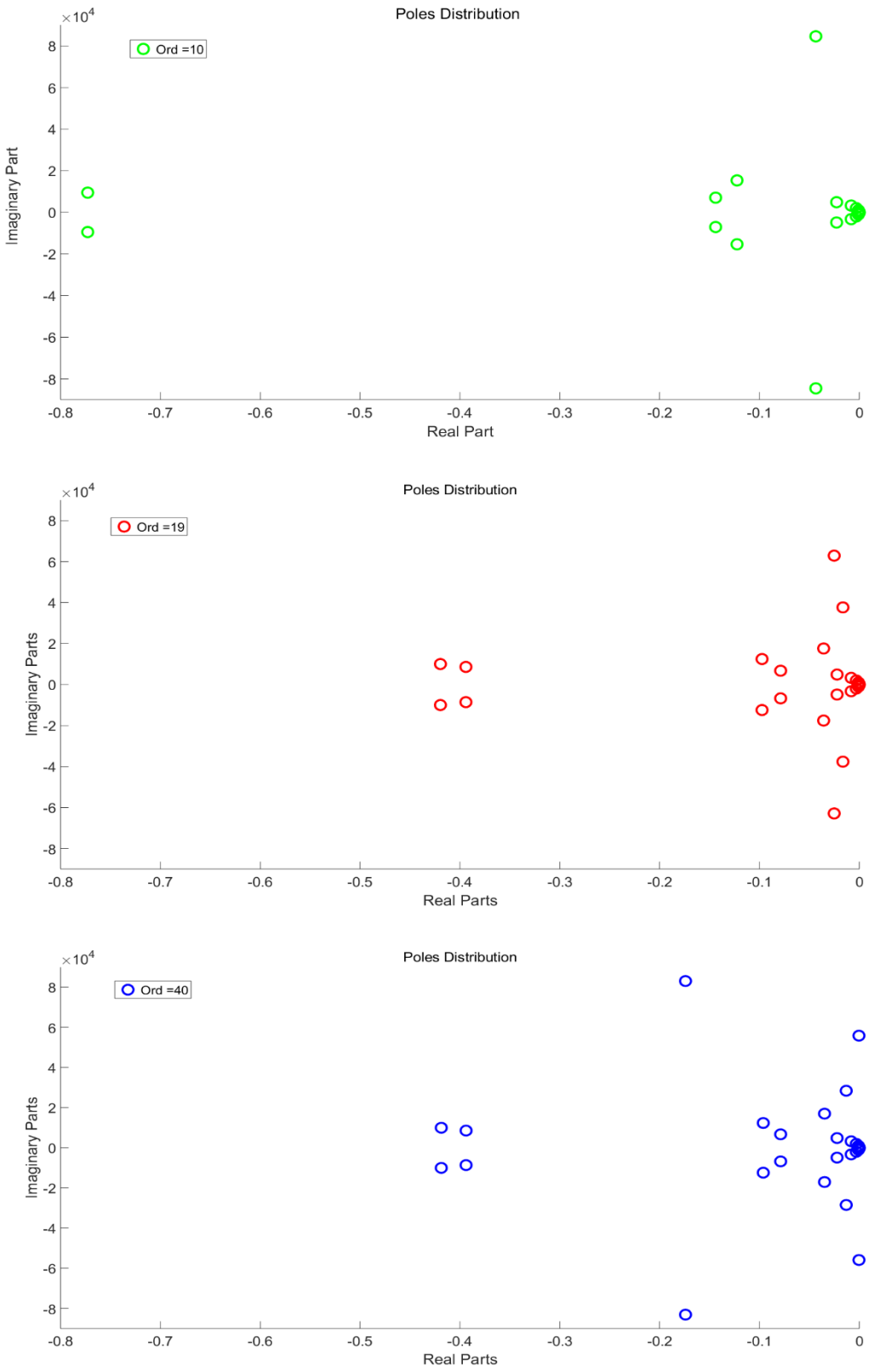


Figure 4.3 Poles distribution of Reduced Models - shaft on bearing support Example –

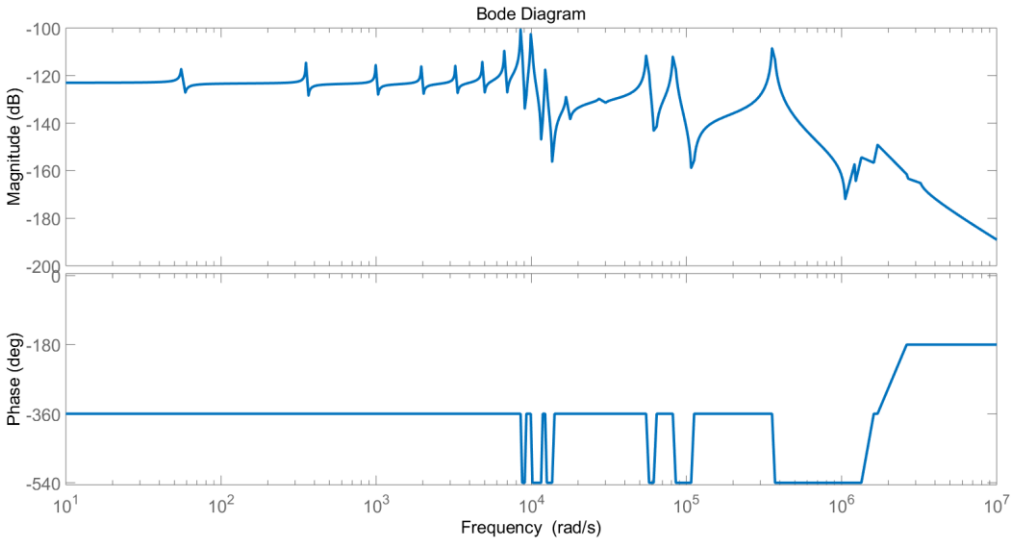


Figure 4.4 Bode Diagram of the reduced model with n = 40 - shaft on bearing support Example –

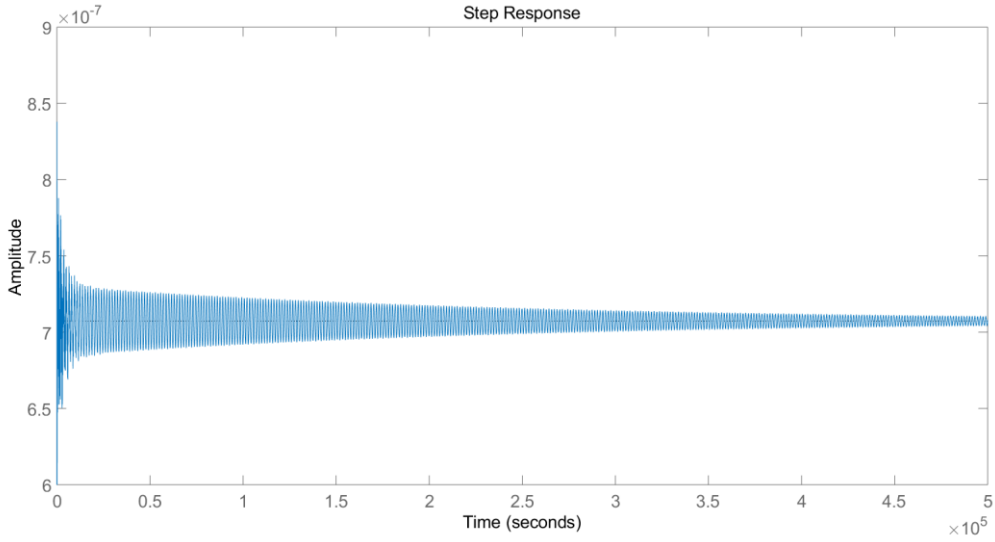


Figure 4.5 Step response of the reduced model with n = 39 - shaft on bearing support Example

IV.5 The butterfly gyroscope

The butterfly gyroscope is MEMS system, which is a vibrating micro-mechanical gyro. The gyro chip consists of a three-layer wafer stack, in which the middle layer contains sensors, the data of this example can be found in the Oberwolfach collection [77]. The order of the full second order system $\sum_N$ is N = 17361, this is single input multiple output SIMO system with one input vector and 12 output vector. The matrices **M** and K are both symmetric and the proportional Rayleigh damping matrix is defined by D = αM + βK. The frequency responses through the 1st output vector of the original and the reduced transfer functions in the frequency range from 10^3 to 10^6 Hz are shown in figure.4.6 with the relative errors, we took α=0 and β=10^{-7}, and the expansion point was equal to $1.05x10^5$. Figure.4.7 shows their poles distribution the stability preservation for the reduced models. Figure 4.8 and 4.9, show the Bode diagram and the step response for the reduced order model (n=39) with the proposed method.

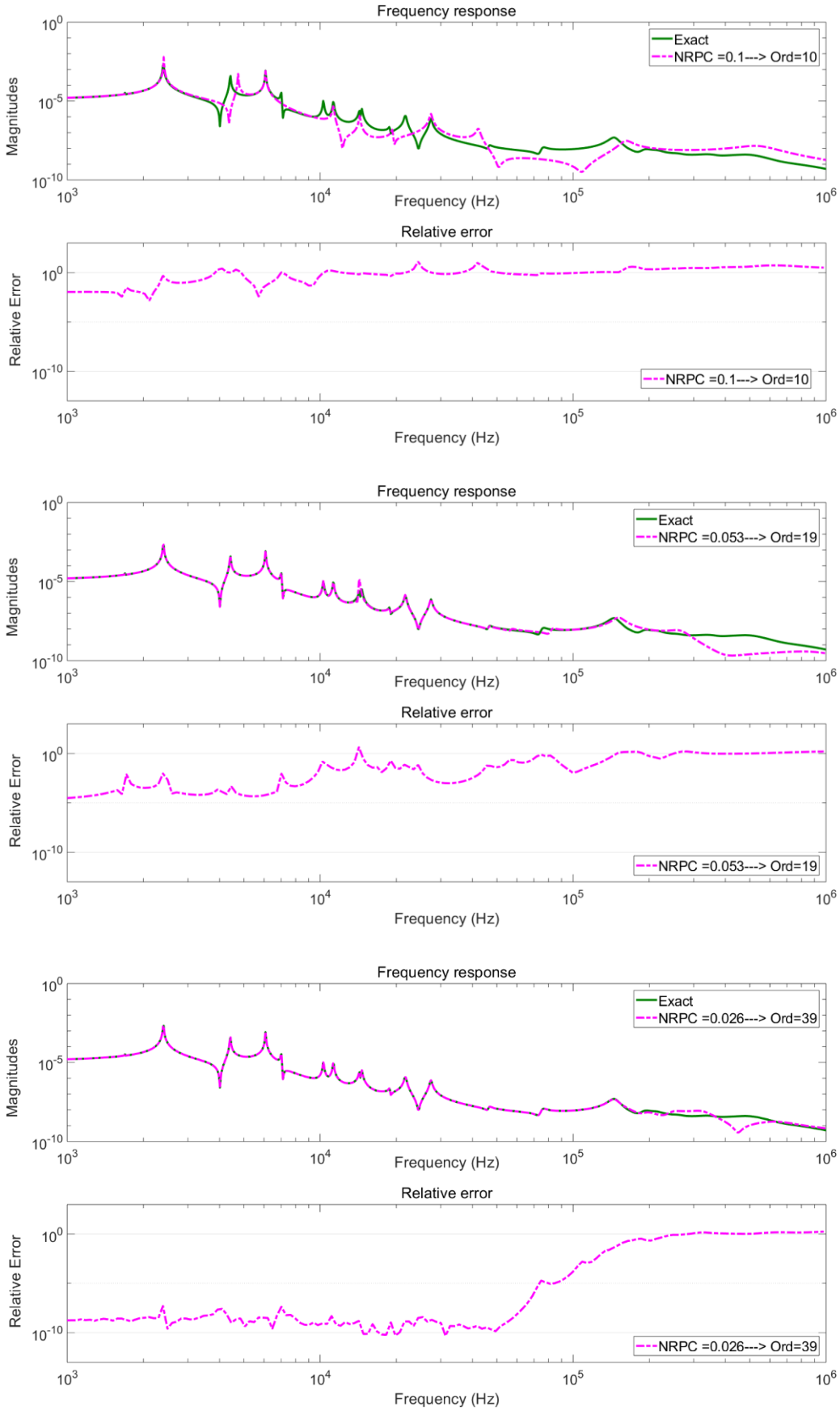


Figure 4.6 Frequency response the relative errors for the butterfly gyroscope

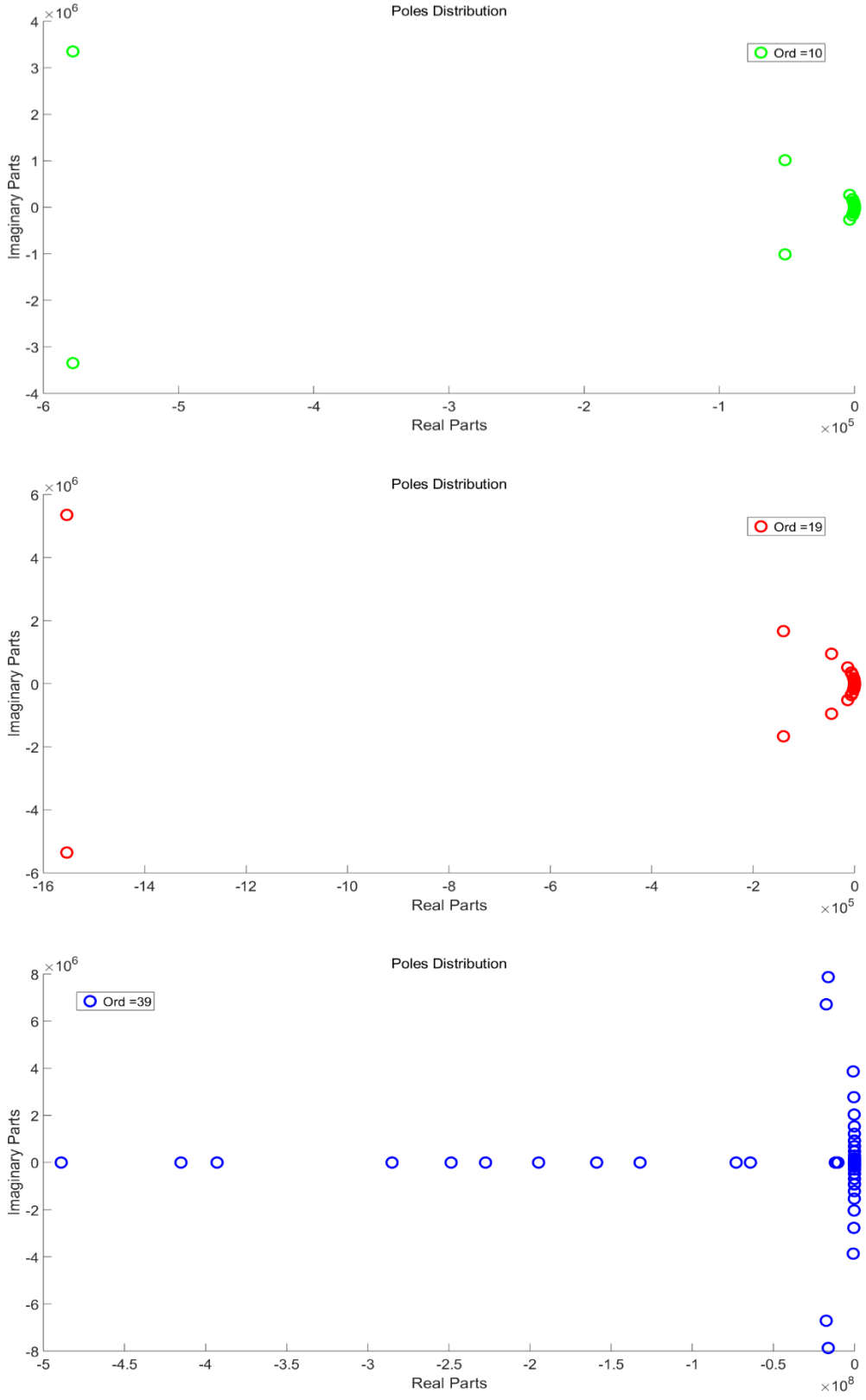


Figure 4.7 Poles distribution of Reduced Models – Butterfly Gyroscope Example –

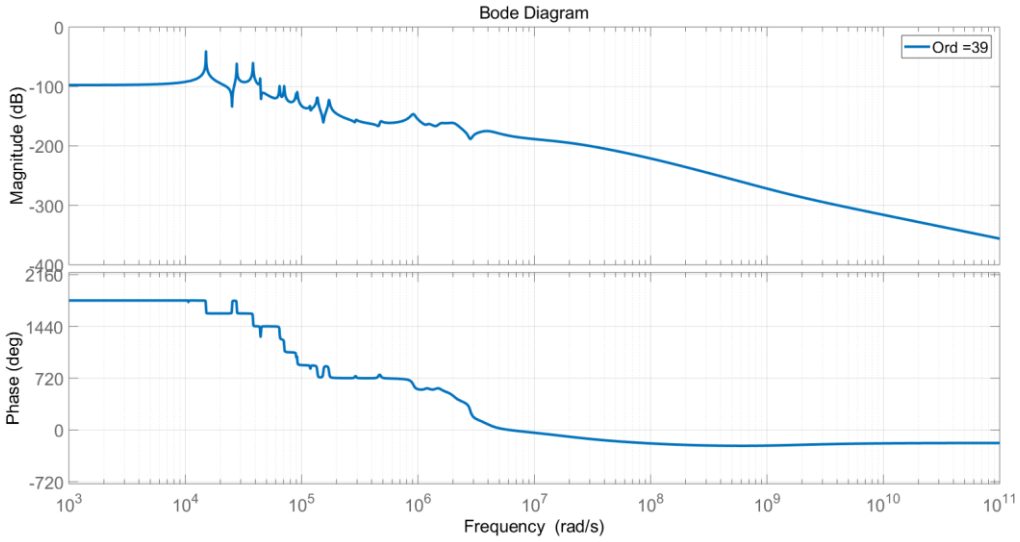


Figure 4.8 Bode Diagram of the reduced model with n = 39 – Butterfly gyroscope Example –

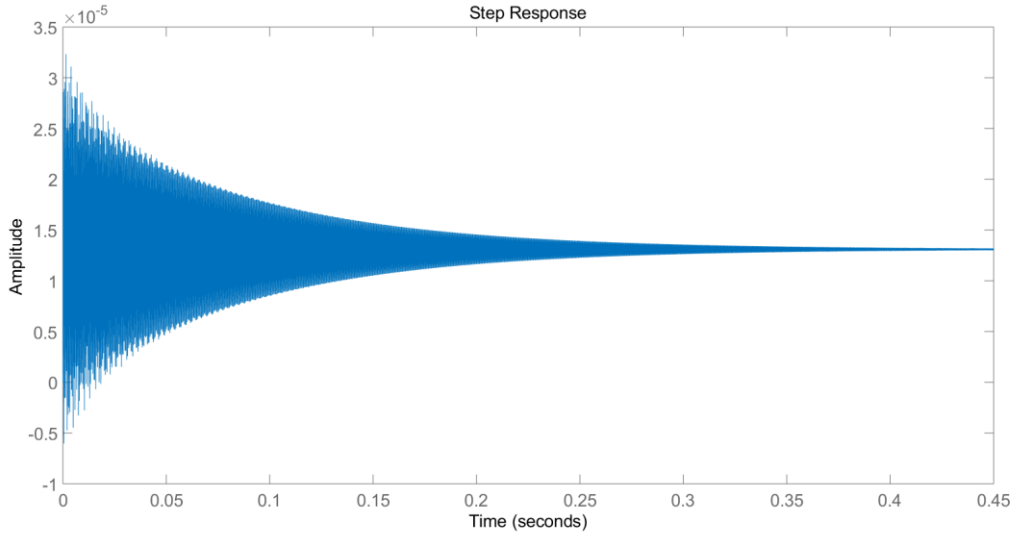


Figure 4.9 Step response of the reduced model with n = 39 – Butterfly gyroscope Example-

IV.6 The 3D Cantilever Timoshenko beam

This is another FE model of 3D Cantilever Timoshenko beam in [78]. The model describes the dynamic behavior of a beam of length equal to 1m where a vertical force $\vec{F}$ is applied at its tip, the dimension of N= 600 for the full second order model $\sum_N$. In the Matlab code in [60] we set the number of elements parameter to 100 and we took s_0=0 as an expansion point. The matrices **M** and **K** are both symmetric and the proportional Rayleigh damping matrix is defined by D = αM + βK. The frequency responses for this SISO (Single Input Single Output) system of the original and the reduced transfer functions in the frequency range from 0 to 1200 Hz are shown in figure.5.10, we took α=8 and β=8*10^{-6}. Figure.4.10 also shows the relative errors for reduced models with order n= 10, 19,39, and Figure.4.11 shows their poles distribution and the stability preservation for the reduced models. Figure 4.12 and 4.13 show the Bode diagram and the step response for the reduced order model (n=39) with the proposed method.

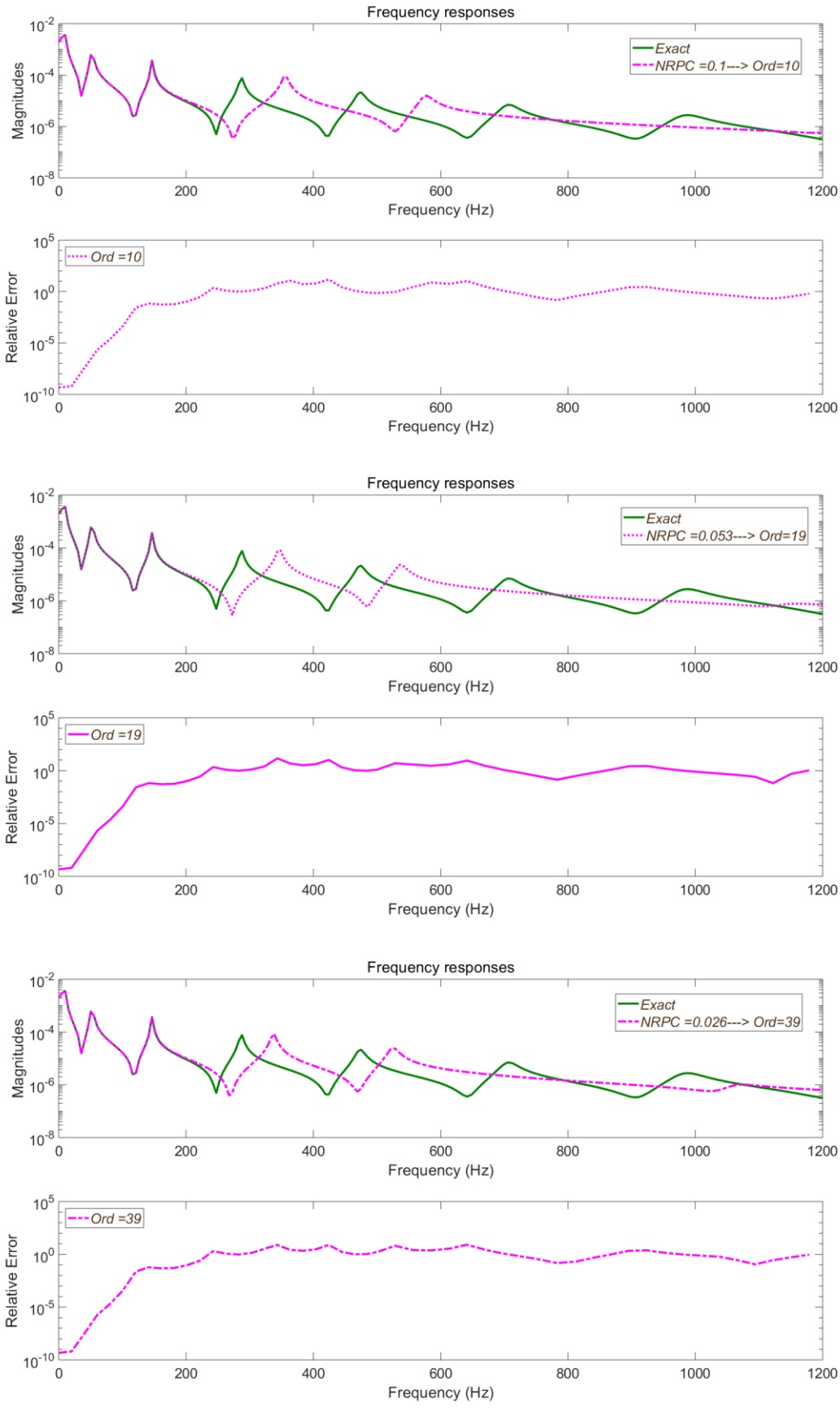


Figure 4.10 Frequency response the relative errors for the 3D Cantilever Timoshenko beam

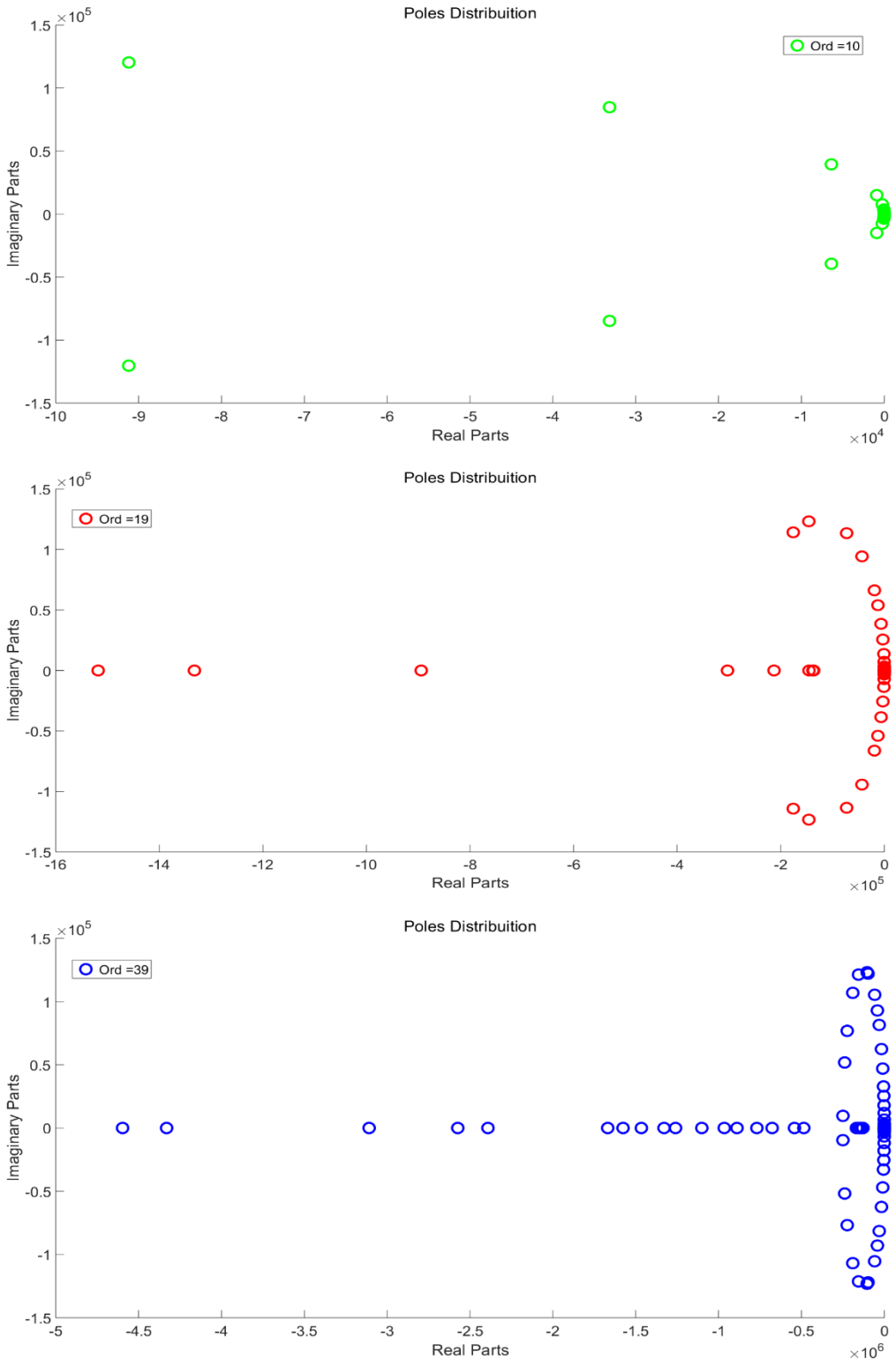


Figure 4.11 Poles distribution of Reduced Models – 3D Cantilever Timoshenko beam –

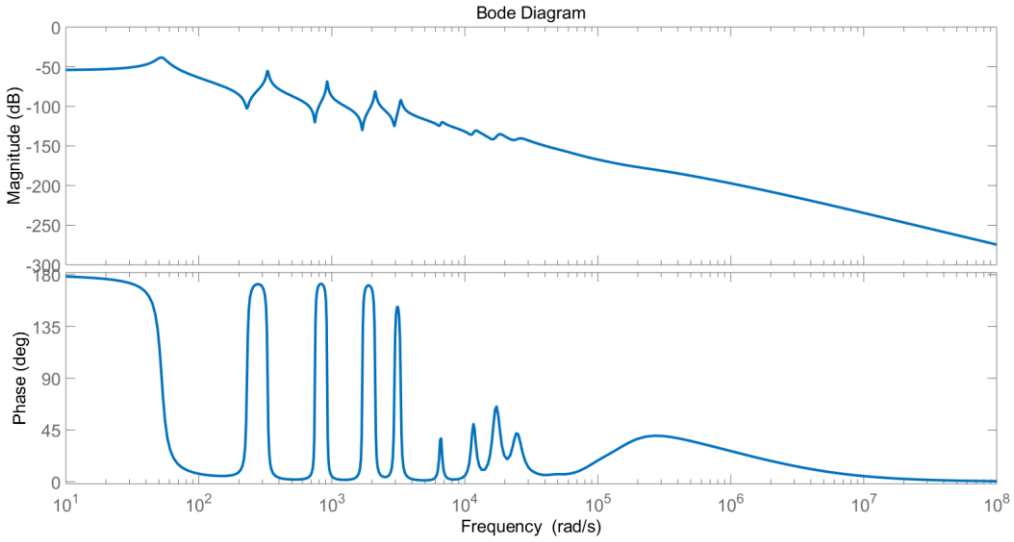


Figure 4.12 Bode Diagram of the reduced model with n = 39 – 3D Cantilever Timoshenko beam –

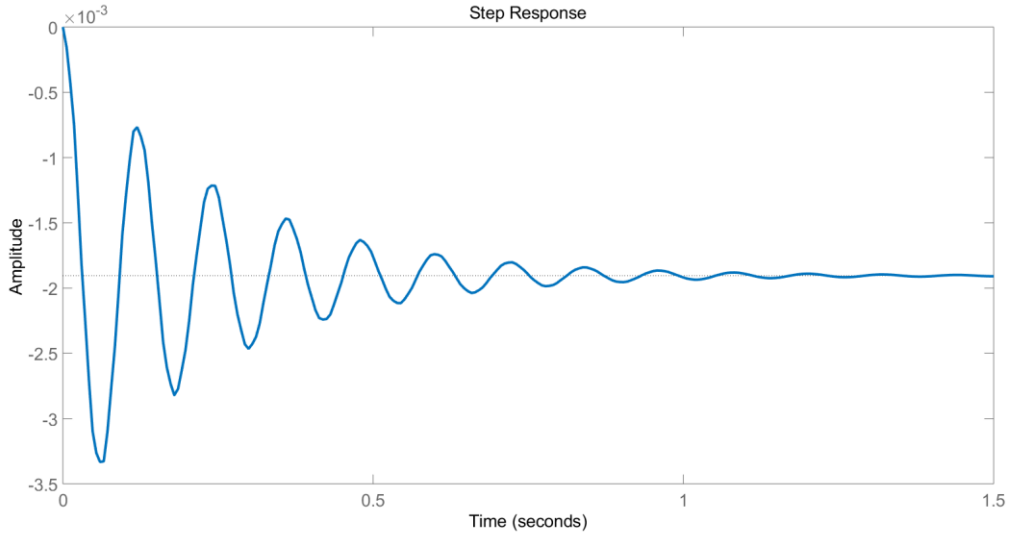


Figure 4.13 Step response of the reduced model with n = 39 – 3D Cantilever Timoshenko beam –

IV.7 Stability analysis

From poles distribution of each ROMs, it is perceivable that all the ROMs computed by the proposed technique of the considering examples are stable (all real parts are negatives).

Figures 4.5, 4.9 and 4.13 exhibit the step-response of the ROMs of the target models originated from the proposed Algorithm. The horizontal axis depicts the time required for the step response to be converged to the equilibrium, whereas the vertical axis is for the amplitude. The ROM of Butterfly Gyroscope converged within a fraction of a second, and the ROMs of the shaft on bearing supports and the 3D Cantilever Timoshenko beam need 5 x 10^5, 1.5 seconds respectively to be converged.

IV.8 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

Table 4.2 represents the $\mathcal{H}_\infty\ norm$, and $\mathcal{H}_2\ norm$ and the steady state error (SS error) of the ROMs achieved by our newly developed technique for the value of NRPC = 0.026 with computing time (denoted CPU) in seconds for the three examples. The results show the convergence behavior of the proposed method with an acceptable error in short lapse of time.

Norms / *ROMs Model*	$\mathcal{H}_2$	$\mathcal{H}_\infty$	***SS error***	***CPU***
shaft on bearing support (*n= 40)*	*0.3308*	*0.3230*	*0.068493*	*0.5635*
Butterfly gyroscope (*n =39)*	*1.3312E-05*	*3.5104E-06*	*1.31E-005*	*2.8540*
3D Cantilever Timoshenko beam (*n= 39)*	*0.0224*	*0.0143*	*0.00191*	*0.3099*

Table 4.2 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms and the SS error of the ROMs

IV.9 Conclusion

Based on the obtained results, the proposed algorithm showed through the three numerical examples, high efficiency and accuracy in terms of relative error against the original systems, however the keys properties of the 2nd order form in the reduced model are still preserved, also proved its superiority compared to the conventional SOAR in terms of robustness, where a suitable and optimal reduced order has been chosen systematically.

The proposed approach works very well for the three examples used in numerical tests (FE Model of a shaft on bearing supports with a damper, a butterfly gyroscope model, and the FE Model of a 3D Cantilever Timoshenko Beam) where the key proprieties such as the preservation of the of Second order structure and the stability was guaranteed with an automatic selection of a significant reduced order as a size of the reduced model in the numerical simulation results.

General Conclusion

General Conclusion

In this thesis, we investigate a new technique for the problem of model order reduction for different kinds of mechanical systems described by differential equations involving second-order time derivatives. The contribution described in this thesis are of theoretical as well as computational nature.

In chapter 1 and chapter 2, some basics and fundamentals on LTI systems and techniques from model order reduction theory were presented. A special focus on the model order reduction of second-order models based on the Krylov subspace has been exposed. We have also shown a new idea for reducing a second order linear time invariant system with structure preserving; the proposed model order reduction approach is based on the definition of a new coefficient for an efficient stopping criterion called Numerical Rank Performance Coefficient NRPC and an extension of the general definition of the Krylov sub-space to second-order Krylov sub-space. The NRPC was used to find the suitable order of the reduced model, and this by using a Second Order ARnoldi algorithm as a Krylov MOR approach with auto selection of the reduced order.

In practical applications, often only limited ranges in frequency or time domain are of interest, the reduced order models obtained with proposed technique are compared to the original ones. They indicate that the proposed approach with the automatic stopping criteria exhibit better performance in all experiments and provide the best optimal solution during the search mechanism with preservation of the second-order structure and the propriety of stability.

The contribution of this work are summarized below:

1. This work is the first to reduce a second-order system directly from the second-order form of the original model with auto-selection of the reduced order.
2. A computationally attractive and analytically simple model reduction approach based on Second-order Krylov sub-space is introduced.
3. A stable and improved reduced order models are obtained from benchmark model reduction problems.
4. The results of this work have been published in international journal.

In conclusion, these new ideas presented above opened the door to new methods that could be applied to other algorithms, such as SVD-based algorithms or evolutionary algorithms.

Some possible future research area in model reduction could be finding a relationship between the NRPC coefficient and the relative error to define a global bounds the global error. Another possible research area would be to the study $\mathcal{H}_\infty\ norm$, and $\mathcal{H}_2\ norm$ of the reduced model with some constraints on the norms, steady state error, or any other aspect of the system.

References

Refrences

1. Salim R, Bettayeb M (2009) H2 optimal model reduction using genetic algorithms and particle swarm optimization. *2009 6th Int Symp Mechatronics its Appl ISMA 2009*.
2. Bettayeb M (1981) Approximation of Linear Systems: New Approaches based on Singular.
3. Moore B (1981) Principal component analysis in linear systems: controllability, observability,. *IEEE Trans Autom Control* 26: 17–32.
4. Glover. K (1984) All optimal hankel-norm approximations of linear multivariable systems,. *Int J Control* 39: 1115–1193.
5. Bai. Z (2002) Krylov subspace techniques for reduced-order modeling of large-scale dynamical,. *Appl Numer Math* 43: 9–44.
6. Hung-Chi Tsai b, Ming-Hong Lai c C-CC (2010) Structure preserving model-order reductions of MIMO second-order systems using Arnoldi methods,. *Math Comput Model* 51: 956–973.
7. Su. ZB and Y (2006) Dimension reduction of large-scale second-order dynamical systems via asecond-order Arnoldi method. *SIAM J Sci Comput* 26: 1692–1709.
8. A. C. Antoulas (2005) Approximation of Large-Scale Dynamical Systems, Philadelphia, PA, SIAM Publications.
9. A. C. Antoulas, C. A. Beattie and SG (2020) Interpolatory Methods for Model Reduction, Philadelphia, PA, Society for Industrial and Applied Mathematics,.
10. P. Benner, A. Cohen, M. Ohlberger and KW (2017) Model Reduction and Approximation: Theory and Algorithms, Philadelphia, PA, Computational Science & Engineering, SIAM Publications,.
11. Pritchard DH and AJ (2005) Mathematical Systems Theory I: Modelling, State Space Analysis, Stability and Robustness, Berlin/Heidelberg, Germany, Springer-Verlag,.
12. C. JWP and J (1998) Introduction to Mathematical Systems Theory: A Behavioral Approach, New York, NY, USA, Springer-Verlag,.
13. Sontag ED (1998) Mathematical Control Theory: Deterministic Finite Dimensional Systems, New York, NY, USA, Springer-Verlag,.
14. Gugercin CAB and S (2009) Interpolatory projection methods for structurepreserving model reduction,. *Syst Control Lett* 58: 225–232.
15. G. M. Flagg, C. A. Beattie and SG (2013) Interpolatory H∞ model reduction,. *Syst Control Lett* 62: 567–574.
16. H. Panzer, T. Wolf and BL (2012) A strictly dissipative state space representation of second order systems,. *at-Automatisierungstechnik*, 60: 392–397.
17. Stykel TR and T (2008) Balanced truncation model reduction of second-order systems,. *Math Comput Model Dyn Syst* 14: 391–406.

18. T. Bonin, H. Faßbender, A. Soppa and MZ (2016) A fully adaptive rational global Arnoldi method for the model-order reduction of second-order MIMO systems with proportional damping,. *Math Comput Simulat* 122: 1–19.

19. Collection OB (2004) Butterfly gyroscope. hosted at MORwiki – Model Order Reduction Wiki, 2004. Available from: http://modelreduction.org/index.php/Butterfly_Gyroscope.

20. E. Kelasidi, P. Liljebäck, K. Y. Pettersen and JG (2016) Innovation in underwater robots: Biologically inspired swimming snake robots,. *IEEE Robot Autom Mag* 23: 44–62.

21. Fish F (2013) Advantages of natural propulsive systems,. *Mar Technol Soc Journal*, 47: 37–44.

22. J. Saak, D. Siebelts and SWRW (2019) A comparison of second-order model order reduction methods for an artificial fishtail,. *at-Automatisierungstechnik* 67: 648–667.

23. D. Siebelts, A. Kater and TM (2018) Modeling and motion planning for an artificial fishtail,. *IFAC-PapersOnLine*, 51: 319–324.

24. D. Siebelts, A. Kater, T. Meurer and JA (2019) hosted at MORwiki – Model Order Reduction Wiki, Matrices for an artificial fishtail., 2019.

25. A. D. Marchese, C. D. Onal and DR (2014) Autonomous soft robotic fish capable of escape maneuvers using fluidic elastomer actuators,. *Soft Robot* 1: 75–87.

26. Wilhelmus H.A. Schilders, Henk A. van der Vorst JR (2008) Model Order Reduction:Theory, Research Aspects and Applications, Verlag Berlin Heidelberg, Springer.

27. Tan S, He L (2007) Advanced model order reduction techniques in VLSI design, Cambridge University Press.

28. Kailath T (1980) Linear systems, Prentice-Hall Englewood Cliffs, NJ.

29. Ahmad MI (2011) Krylov subspace techniques for model reduction and the solution of linear matrix equations.

30. Bai Z-J, Su Y-F (2004) Second-Order Krylov Subspace and Arnoldi Procedure.

31. Bai Z, Yangfeng SU (2005) Dimension reduction of large-scale second-order dynamical systems via a second-order arnoldi method. *SIAM J Sci Comput* 26: 1692–1709.

32. Boley DL (1994) KRYLOV SPACE METHODS ON STATE-SPACE CONTROL MODELS*.

33. Bonotto M, Cenedese A, Bettini P (2017) Krylov Subspace Methods for Model Order Reduction in Computational Electromagnetics, *IFAC-PapersOnLine*, Elsevier B.V., 6355–6360.

34. Druskin V, Simoncini V (2011) Adaptive rational Krylov subspaces for large-scale dynamical systems. *Syst Control Lett* 60: 546–560.

35. Eric James Grimme (University of illinois) (1992) Krylov Projection Methods for Model Reduction. *Unirsity of illinois*.

36. Frommer A, Jacob B, Kahl K, et al. (2019) Krylov type methods exploiting the quadratic numerical range.

37. Frommer A, Jacob B, Kahl K, et al. (2020) Krylov type methods for linear systems exploiting properties of the quadratic numerical range. *Electron Trans Numer Anal* 53: 541–561.

38. Obinata G, Anderson BDO (2012) Model reduction for control system design, Springer Science & Business Media.

39. Benner P, Mehrmann V, Sorensen DC (2005) Dimension reduction of large-scale systems, Springer.

40. Antoulas AC, Sorensen DC, Gugercin S (2000) A survey of model reduction methods for large-scale systems.

41. Saad Y (2003) Iterative methods for sparse linear systems, SIAM.

42. Benner P (2011) Partial stabilization of descriptor systems using spectral projectors, *Numerical Linear Algebra in Signals, Systems and Control*, Springer, 55–76.

43. Bai Z, Li R, Su Y (2008) A unified Krylov projection framework for structure-preserving model reduction, *Model order reduction: Theory, research aspects and applications*, Springer, 75–93.

44. Freund RW (2008) Structure-preserving model order reduction of RCL circuit equations, *Model Order Reduction: Theory, Research Aspects and Applications*, Springer, 49–73.

45. Spescha D, Weikert S, Retka S, et al. (2018) Krylov and modal subspace based model order reduction with A-priori error estimation.

46. Nouri B, Nakhla MS, Deng X (2017) Stable model-order reduction of active circuits. *IEEE Trans Components, Packag Manuf Technol* 7: 710–719.

47. Ghosh S, Senroy N (2012) A comparative study of two model order reduction approaches for application in power systems, *2012 IEEE Power and Energy Society General Meeting*, IEEE, 1–8.

48. Fortuna L, Nunnari G, Gallo A (2012) Model order reduction techniques with applications in electrical engineering, Springer Science & Business Media.

49. Chaniotis D, Pai MA (2005) Model reduction in power systems using Krylov subspace methods. *IEEE Trans Power Syst* 20: 888–894.

50. Ionutiu R, Rommes J (2011) On synthesis of reduced order models, *Model reduction for circuit simulation*, Springer, 207–223.

51. Green M, Limebeer DJN (2012) Linear robust control, Courier Corporation.

52. Golub GH, Van Loan CF (2013) Matrix computations, JHU press.

53. Rommes J, Martins N (2006) Efficient computation of multivariable transfer function dominant poles using subspace acceleration. *IEEE Trans power Syst* 21: 1471–1483.

54. Rommes J (2007) Methods for eigenvalue problems with applications in model order reduction.

55. Ionutiu R, Rommes J (2009) Model order reduction for multi-terminal circuits. *COMSON Handb*.

56. Lutowska A (2012) Model order reduction for coupled systems using low-rank approximations.

57. Lohmann B, Salimbahrami B (2000) Introduction to Krylov subspace methods in model order reduction. *Methods Appl Autom* 1–13.

58. Freund RW (2000) Krylov-subspace methods for reduced-order modeling in circuit simulation. *J Comput Appl Math* 123: 395–421.

59. Cullum J, Zhang T (2002) Two-sided Arnoldi and nonsymmetric Lanczos algorithms. *SIAM J Matrix Anal Appl* 24: 303–319.

60. Datta B (2004) Numerical methods for linear control systems, Academic Press.

61. Bai Z, Su Y (2005) Dimension reduction of large-scale second-order dynamical systems via a second-order Arnoldi method. *SIAM J Sci Comput* 26: 1692–1709.

62. Chu C-C, Tsai H-C, Lai M-H (2010) Structure preserving model-order reductions of MIMO second-order systems using Arnoldi methods. *Math Comput Model* 51: 956–973.

63. Su Y, Wang J, Zeng X, et al. (2004) SAPOR: Second-order Arnoldi method for passive order reduction of RCS circuits, *IEEE/ACM International Conference on Computer Aided Design, 2004. ICCAD-2004.*, IEEE, 74–79.

64. Yang C (2005) Solving large-scale eigenvalue problems in SciDAC applications, *Journal of Physics: Conference Series*, IOP Publishing, 58.

65. Puri RS, Morrey D (2013) A comparison of one-and two-sided Krylov–Arnoldi projection methods for fully coupled, damped structural-acoustic analysis. *J Comput Acoust* 21: 1350004.

66. Puri RS (2009) Krylov subspace based direct projection techniques for low frequency, fully coupled, structural acoustic analysis and optimization.

67. Su T-J, Craig Jr RR (1991) Model reduction and control of flexible structures using Krylov vectors. *J Guid Control Dyn* 14: 260–267.

68. Bai Z, Su Y (2005) SOAR: A second-order Arnoldi method for the solution of the quadratic eigenvalue problem. *SIAM J Matrix Anal Appl* 26: 640–659.

69. Zhou L, Bao L, Lin Y, et al. (2010) Restarted generalized second-order krylov subspace methods for solving quadratic eigenvalue problems. *Int J Math Comput Sci* 4: 997–1004.

70. Beattie CA, Gugercin S (2005) Krylov-based model reduction of second-order systems with proportional damping, *Proceedings of the 44th IEEE Conference on Decision and Control*, IEEE, 2278–2283.

71. Salimbahrami B, Lohmann B (2006) Order reduction of large scale second-order systems using Krylov subspace methods. *Linear Algebra Appl* 415: 385–405.

72. Vakilzadeh M, Eghtesad M, Vatankhah R, et al. (2018) Model order reduction of second-order systems with nonlinear stiffness using Krylov subspace methods and their

symmetric transfer functions. *Int J Syst Sci* 49: 2630–2643.

73. Uddin MM (2020) Structure preserving model order reduction of a class of second-order descriptor systems via balanced truncation. *Appl Numer Math* 152: 185–198.

74. Salimbahrami B, Lohmann B, Bechtold T, et al. (2005) A two-sided Arnoldi algorithm with stopping criterion and MIMO selection procedure. *Math Comput Model Dyn Syst* 11: 79–93.

75. Tamri A, Mitiche L, Adamou-Mitiche ABH (2022) A Second Order Arnoldi Method with Stopping Criterion and Reduced Order Selection for Reducing Second Order Systems. *Eng Technol Appl Sci Res* 12: 8712–8717.

76. Kowalski TR (2000) Extracting a few eigenpairs of symmetric indefinite matrix pencils, University of Kentucky.

77. Korvink JG, Rudnyi EB (2005) Oberwolfach benchmark collection, *Dimension reduction of large-scale systems*, Springer, 311–315.

78. Panzer H, Hubele J, Eid R, et al. (2009) Generating a parametric finite element model of a 3d cantilever timoshenko beam using matlab, Lehrstuhl für Regelungstechnik.